

Frame transforms, star products and quantum mechanics on phase space

This article has been downloaded from IOPscience. Please scroll down to see the full text article.

2008 J. Phys. A: Math. Theor. 41 285304

(<http://iopscience.iop.org/1751-8121/41/28/285304>)

View [the table of contents for this issue](#), or go to the [journal homepage](#) for more

Download details:

IP Address: 171.66.16.149

The article was downloaded on 03/06/2010 at 06:59

Please note that [terms and conditions apply](#).

Frame transforms, star products and quantum mechanics on phase space

P Aniello^{1,2}, V I Man'ko³ and G Marmo¹

¹ Dipartimento di Scienze Fisiche dell'Università di Napoli 'Federico II' and Istituto Nazionale di Fisica Nucleare (INFN)-Sezione di Napoli, Complesso Universitario di Monte S. Angelo, via Cintia, 80126 Napoli, Italy

² Facoltà di Scienze Biotecnologiche, Università di Napoli 'Federico II', Napoli, Italy

³ P.N. Lebedev Physical Institute, Leninskii Prospect 53, Moscow 119991, Russia

E-mail: paolo.aniello@na.infn.it, manko@na.infn.it and marmo@na.infn.it

Received 11 April 2008, in final form 26 May 2008

Published 19 June 2008

Online at stacks.iop.org/JPhysA/41/285304

Abstract

Using the notions of frame transform and of square integrable projective representation of a locally compact group G , we introduce a class of isometries (tight frame transforms) from the space of Hilbert–Schmidt operators in the carrier Hilbert space of the representation into the space of square integrable functions on the direct product group $G \times G$. These transforms have remarkable properties. In particular, their ranges are reproducing kernel Hilbert spaces endowed with a suitable 'star product' which mimics, at the level of functions, the original product of operators. A 'phase space formulation' of quantum mechanics relying on the frame transforms introduced in the present paper, and the link of these maps with both the Wigner transform and the wavelet transform are discussed.

PACS numbers: 03.65.Ca, 03.65.Ta, 03.65.Wj, 02.20.Qs

1. Introduction

The formulation of quantum mechanics 'on phase space' dates back to the early stages of development of quantum theory. As is well known, the foundations of this elegant formulation have been laid by E Wigner in his 1932 celebrated paper [1], with the aim of exploring the quantum corrections to classical statistical mechanics. Strictly related to Wigner's work are the pioneering studies of H Weyl on quantization [2]. On one hand, Wigner was interested in associating with a quantum state a suitable phase space 'quasi-probability distribution' (association that leads to the Wigner transform). On the other hand, Weyl aimed at associating with a classical observable—a function on phase space—a quantum observable in such a way to overcome the ambiguities related to the 'operator ordering' (association that leads to the

Weyl map). These procedures can be regarded as the two ‘arrows’ of a unique theoretical framework that we may call the ‘quantization–dequantization theory’. This subject is a richly branched, old—but still extremely vital—tree. Since it is really huge, we will not attempt at giving even a brief overview; the reader may consult the collection of papers [3] (and the bibliography therein) as a general reference on the subject.

It is also worth mentioning the fact that, quite recently, the impressive progress of experimental techniques—as well as the need of gaining a deeper understanding of some fundamental (and controversial) aspects of quantum mechanics—have motivated a renewed interest in the description of quantum states by means of phase space functions, the so-called ‘quantum state tomography’ or simply ‘quantum tomography’; see, e.g. [4–9].

There is a deep link between the quantization–dequantization theory (including the formalism of quantum tomography) and another huge research area—mainly focused on applications to signal analysis—which we may globally call ‘(generalized) wavelet analysis’. The main mathematical tool in wavelet analysis is that of *frame* [10], a notion that will play a central role in the present paper. Again, we will make no attempt at providing an overview on this vast and interesting subject; we will then refer the reader to the excellent references [11–14]. It is a remarkable fact that several issues, concepts and techniques can be translated ‘from one language into another’—from quantum theory into signal analysis and vice versa—opening the way to new insights (see, e.g. [15]). Several anticipations of the unified framework encompassing the quantization–dequantization theory and wavelet analysis were already present in the pioneering work of Klauder (and his co-authors), who introduced a ‘continuous representation theory’ [16, 17], and of Cahill and Glauber [18].

It turns out that, from the mathematical point of view, the main *trait d’union* between the two mentioned subjects is the remarkable notion of *square integrable representation* [19–22]. In fact, using this invaluable mathematical tool, one is able to perform all the fundamental tasks of the quantization–dequantization theory and of generalized wavelet analysis:

- to define generalized families of coherent states (covariant frames), see [11, 12, 23]; in particular, the standard family of coherent states of Schrödinger [24], Glauber [25], Klauder [16] and Sudarshan [26];
- to obtain ‘discretized frames’ from the covariant frames; see, e.g. [27, 28];
- to define suitable—in the manner of Weyl–Wigner—quantization–dequantization maps; see, e.g. [11, 12, 29, 30].

The aim of the present paper is to reconsider the previously mentioned link between the quantization–dequantization theory and the generalized wavelet analysis. In fact, we believe that to a renewed interest in this area of research should correspond a renewed study of its conceptual and mathematical foundations. As we will try to show, this study leads, in a quite natural way, to the definition of a certain class of ‘frame transforms’ associated with square integrable representations. These transforms are isometries mapping a space of Hilbert–Schmidt operators (which is, obviously, a Hilbert space) onto a space of square integrable functions having remarkable properties. More precisely, given a square integrable projective representation U of a locally compact group G in a Hilbert space \mathcal{H} and a (suitable) Hilbert–Schmidt operator \hat{T} in \mathcal{H} , one can associate with \hat{T} an isometry $\mathfrak{D}_{\hat{T}}$ mapping $\mathcal{B}_2(\mathcal{H})$ (the space of Hilbert–Schmidt operators in \mathcal{H}) into $L^2(G \times G)$ (the Hilbert space of square integrable \mathbb{C} -valued functions on the direct product group $G \times G$, with respect to the left Haar measure). As will be shown, the isometry $\mathfrak{D}_{\hat{T}}$ has remarkable properties that can be regarded as direct consequences of the fact that $\mathfrak{D}_{\hat{T}}$ is a frame transform; in particular:

- (i) The range $\text{Ran}(\mathfrak{D}_{\hat{T}})$ of the isometry $\mathfrak{D}_{\hat{T}}$ is a ‘reproducing kernel Hilbert space’ embedded in $L^2(G \times G)$.

- (ii) The image, through the isometry $\mathfrak{D}_{\hat{\rho}}$, of the product of operators in $\mathcal{B}_2(\mathcal{H})$ is a ‘star product of functions’ in $\text{Ran}(\mathfrak{D}_{\hat{\rho}})$.
- (iii) The standard expectation value formula of quantum mechanics— $\langle \hat{A} \rangle_{\hat{\rho}} = \text{tr}(\hat{A}\hat{\rho})$, where \hat{A} and $\hat{\rho}$ are, respectively, a bounded selfadjoint operator and a density operator in \mathcal{H} —admits, in this framework, a suitable expression in terms of \mathbb{C} -valued functions ‘on phase space’.

The adjoint $\mathfrak{Q}_{\hat{\rho}}$ of the isometry $\mathfrak{D}_{\hat{\rho}}$, like the Weyl map, has a simple integral expression and can be regarded as a ‘quantization map’.

The paper is organized as follows. In section 2, we discuss the notion of ‘frame transform’ and its main consequences. In section 3, we briefly review the definition of the Wigner distribution and its relation with projective representations. Next, in section 4, we recall the basic properties of square integrable projective representations, tools that are fundamental for the definition of the (generalized) Wigner transform and of its reverse arrow, the (generalized) Weyl map, see section 5; we will also argue that the generalized Wigner transform is not, in general, a frame transform. Our analysis will culminate in the introduction of the class of frame transforms mentioned before—section 6—and in the discussion of the main consequences from the point of view of quantum mechanics, see section 7. In section 8, we consider a remarkable example that allows us to show the link of our results with the formalism of s -parametrized quasi-distributions developed by Cahill and Glauber [18]. Eventually, in section 9, a few conclusions are drawn.

2. Frame transforms and star products

In this section, we will introduce the mathematical notions of ‘frame’ and of ‘frame transform’ that will be central in the following. In particular—in the present section and later, on the base of our main results, in section 7—we will show that by means of these notions it is possible, in a natural way, to define a class of ‘star products’ of functions and to introduce a formulation of quantum mechanics ‘on phase space’. In the first part of the section, we will collect a few basic facts on frames in Hilbert spaces, a subject which is discussed with plenty of applications in several excellent references; see, e.g. [13, 14]. In the second part of the section, we will focus on the peculiar case of frames in Hilbert–Schmidt spaces (of operators). As we will show, in this case the theory of frames enjoys extra results reflecting the fact that a space of Hilbert–Schmidt operators is not only a Hilbert space but is also endowed with the structure of an algebra.

Let \mathcal{S} be a separable complex Hilbert space (we will denote by $\langle \cdot, \cdot \rangle$ the associated scalar product) and $\mathcal{X} = (X, \mu)$ a measure space. A family of vectors \mathcal{S}_X in \mathcal{S} , labeled by points in X , $\mathcal{S}_X = \{\psi_x \in \mathcal{S} : x \in X\}$, is called a *frame* (in \mathcal{S} , based on the measure space \mathcal{X}) if it satisfies the following defining conditions:

- For every $\phi \in \mathcal{S}$, the function

$$\Phi : X \ni x \mapsto \langle \psi_x, \phi \rangle \in \mathbb{C} \tag{1}$$

is μ -measurable and belongs to $L^2(X) \equiv L^2(X, \mu; \mathbb{C})$.

- The ‘stability condition’ is verified, namely,

$$\alpha \|\phi\|_{\mathcal{S}}^2 \leq \|\Phi\|_{L^2(X)}^2 = \int_X |\Phi(x)|^2 d\mu(x) \leq \beta \|\phi\|_{\mathcal{S}}^2, \quad \forall \phi \in \mathcal{S}, \tag{2}$$

for some (fixed) $\alpha, \beta \in \mathbb{R}$ such that $0 < \alpha \leq \beta$.

A couple of strictly positive numbers α, β —such the the stability condition (2) is satisfied—are called (lower and upper) *frame bounds* for the frame \mathcal{S}_X ; in particular, the frame \mathcal{S}_X is said to be *tight* if one can set $\alpha = \beta$.

Therefore, a frame $\mathcal{S}_X = \{\psi_x\}_{x \in X}$ defines a *frame transform* (operator), i.e. the linear operator

$$\mathfrak{F} : \mathcal{S} \ni \phi \mapsto \Phi := \langle \psi_{(\cdot)}, \phi \rangle \in L^2(X), \tag{3}$$

which is bounded

$$\|\mathfrak{F}\phi\|_{L^2(X)}^2 \leq \beta \|\phi\|_{\mathcal{S}}^2, \quad \forall \phi \in \mathcal{S}, \tag{4}$$

injective, and admits a (in general, non-unique) bounded left inverse

$$\alpha \|\phi\|_{\mathcal{S}}^2 \leq \|\mathfrak{F}\phi\|_{L^2(X)}^2, \quad \forall \phi \in \mathcal{S}. \tag{5}$$

For every $\phi \in \mathcal{S}$, the \mathbb{C} -valued function $\mathfrak{F}\phi$ will be called the *frame transform* of ϕ .

Note that the existence of a bounded left inverse of \mathfrak{F} implies that the range of the frame transform— $\text{Ran}(\mathfrak{F})$ —is closed in $L^2(X)$: $\text{Ran}(\mathfrak{F}) = \overline{\text{Ran}(\mathfrak{F})}$. Specifically, \mathfrak{F} admits a (unique) bounded *pseudo-inverse* $\mathfrak{F}^{\leftarrow} : L^2(X) \rightarrow \mathcal{S}$, which is the linear operator determined by the conditions

$$\mathfrak{F}^{\leftarrow} \mathfrak{F} = I, \quad (\mathfrak{F}^{\leftarrow} \text{ is a left inverse of } \mathfrak{F}) \tag{6}$$

$$\mathfrak{F}^{\leftarrow} \Theta = 0, \quad \forall \Theta \in \text{Ran}(\mathfrak{F})^\perp, \text{ (i.e. } \text{Ker}(\mathfrak{F}^{\leftarrow}) = \text{Ran}(\mathfrak{F})^\perp) \tag{7}$$

with I denoting the identity in \mathcal{S} and $\text{Ran}(\mathfrak{F})^\perp$ the orthogonal complement of the subspace $\text{Ran}(\mathfrak{F})$ of $L^2(X)$. Obviously, in the case where $\text{Ran}(\mathfrak{F}) = L^2(X)$, the pseudo-inverse $\mathfrak{F}^{\leftarrow}$ is nothing but the (bounded) inverse \mathfrak{F}^{-1} . However, we stress that the case where $\text{Ran}(\mathfrak{F}) = L^2(X)$ does not occur in several important examples; typically, $\text{Ran}(\mathfrak{F})$ is a proper subspace of $L^2(X)$ consisting of functions with some regularity property (this happens, for instance, in the case where X is a topological space and the frame map $x \mapsto \psi_x$ is weakly continuous).

It is clear that for the adjoint $\mathfrak{F}^* : L^2(X) \rightarrow \mathcal{S}$ of \mathfrak{F} the following formula holds:

$$\mathfrak{F}^* \Phi = \int_X \Phi(x) \psi_x \, d\mu(x), \quad \forall \Phi \in L^2(X), \tag{8}$$

where the integral (as all the vector-valued or operator-valued integrals henceforth) has to be understood ‘in the weak sense’.

By means of the frame operator \mathfrak{F} and of its adjoint \mathfrak{F}^* , one can define the *metric operator* of the frame \mathcal{S}_X , i.e. the map $\hat{M} := \mathfrak{F}^* \mathfrak{F} : \mathcal{S} \rightarrow \mathcal{S}$, which is a bounded, definite positive linear operator (with a bounded definite positive inverse \hat{M}^{-1}): $\alpha I \leq \hat{M} \leq \beta I$. It is easy to verify, using the defining conditions (6)–(7), that the following relation holds:

$$\mathfrak{F}^{\leftarrow} = \hat{M}^{-1} \mathfrak{F}^*. \tag{9}$$

The metric operator allows us to define the *dual frame* of the frame \mathcal{S}_X , namely, the family of operators

$$\mathcal{S}^X := \{\psi^x \in \mathcal{S} : \psi^x = \hat{M}^{-1} \psi_x, \psi_x \in \mathcal{S}_X\}. \tag{10}$$

We stress that the term ‘dual frame’ is coherent: one can easily show that \mathcal{S}^X is indeed a frame (in \mathcal{S} , based on \mathcal{X}). Note that, if the frame \mathcal{S}_X is tight, then \mathfrak{F} is—possibly up to a positive factor—an isometry, the positive operator \hat{M} is a multiple of the identity, and \mathcal{S}_X coincides with its dual frame \mathcal{S}^X up to, possibly, an irrelevant overall normalization factor; i.e., there is

a strictly positive number r such that $\psi^x = r\psi_x, \forall x \in X$. In particular, we will say that the tight frame \mathcal{S}_X is *normalized* if $r = 1$.

Moreover, it is clear that denoting by $\tilde{\mathfrak{F}}$ the frame transform associated with the frame \mathcal{S}^X , we have

$$\tilde{\mathfrak{F}} = \mathfrak{F}\hat{M}^{-1}; \tag{11}$$

hence, the metric operator associated with the frame \mathcal{S}^X is \hat{M}^{-1} and the dual frame of \mathcal{S}^X is \mathcal{S}_X . From relations (9) and (11) it follows that

$$\mathfrak{F}^{\leftarrow}\Phi = \tilde{\mathfrak{F}}^*\Phi = \int_X \Phi(x)\psi^x \, d\mu(x), \quad \forall \Phi \in L^2(X). \tag{12}$$

If, in particular, the frame \mathcal{S}_X is tight, then the pseudo-inverse $\mathfrak{F}^{\leftarrow}$ coincides—possibly up to a positive factor—with $\tilde{\mathfrak{F}}^*$.

By means of a couple of mutually dual frames \mathcal{S}_X and \mathcal{S}^X , one can write some remarkable formulae. In fact, taking into account formula (12), and using the Dirac notation $|\phi\rangle\langle\psi|\eta \equiv \langle\psi, \eta\rangle\phi, \psi, \phi, \eta \in \mathcal{S}$, we can write the following resolutions of the identity:

$$\begin{aligned} I &= \mathfrak{F}^{\leftarrow}\mathfrak{F} = \int_X |\psi^x\rangle\langle\psi_x| \, d\mu(x) \\ &= \hat{M} \int_X |\psi^x\rangle\langle\psi_x| \, d\mu(x)\hat{M}^{-1} = \int_X |\psi_x\rangle\langle\psi^x| \, d\mu(x); \end{aligned} \tag{13}$$

thus, we have a ‘reconstruction formula’ for the frame transform $\mathfrak{F}\phi$, i.e.

$$\phi = \int_X (\mathfrak{F}\phi)(x)\psi^x \, d\mu(x), \quad \forall \phi \in L^2(X), \tag{14}$$

and an analogous formula for the (dual) frame transform $\tilde{\mathfrak{F}}\phi$. From relations (13) we get immediately

$$\hat{M} = \hat{M} \int_X |\psi^x\rangle\langle\psi_x| \, d\mu(x) = \int_X |\psi_x\rangle\langle\psi_x| \, d\mu(x) \tag{15}$$

and

$$\hat{M}^{-1} = \hat{M}^{-1} \int_X |\psi_x\rangle\langle\psi^x| \, d\mu(x) = \int_X |\psi^x\rangle\langle\psi^x| \, d\mu(x). \tag{16}$$

Moreover, observe that for the orthogonal projection $\hat{P}_{\text{Ran}(\mathfrak{F})}$ onto the subspace $\text{Ran}(\mathfrak{F})$ of $L^2(X)$ we have the following remarkable expression:

$$(\hat{P}_{\text{Ran}(\mathfrak{F})}\Phi)(x) = (\mathfrak{F}\mathfrak{F}^{\leftarrow}\Phi)(x) = \int_X \varkappa(x, x')\Phi(x') \, d\mu(x'), \quad \forall \Phi \in L^2(X), \tag{17}$$

for μ -almost all (in short, for μ -a.a.) $x \in X$, where $\varkappa(\cdot, \cdot)$ is the \mathbb{C} -valued function on $X \times X$ defined by

$$\varkappa(x, x') := \langle\psi_x, \psi^{x'}\rangle, \quad \forall x, x' \in X. \tag{18}$$

Therefore, the range of the frame operator is a *reproducing kernel Hilbert space* (in short, r.k.H.s.) [31–33].

Remark 1. Strictly speaking, $\text{Ran}(\mathfrak{F})$ is a ‘r.k.H.s. embedded in $L^2(X)$ ’. The ‘true’ r.k.H.s. is the vector space composed of every \mathbb{C} -valued function Φ on X of the form $\Phi = \langle\psi_{(\cdot)}, \phi\rangle, \phi \in \mathcal{S}$. Embedding this r.k.H.s. in $L^2(X)$ amounts to identifying such a function Φ with the equivalence class of μ -measurable \mathbb{C} -valued functions on X that coincide with Φ for μ -a.a. $x \in X$, as is tacitly done usually (e.g., in definition (3)).

It is an interesting fact that every bounded operator in the r.k.H.s. $\text{Ran}(\mathfrak{F})$ is an integral operator. Precisely, as the reader may check using formula (12), for every operator \hat{A} in $\mathcal{B}(\mathcal{S})$ (the Banach space of bounded linear operators in \mathcal{S}), we have

$$((\mathfrak{F}\hat{A}\mathfrak{F}^{\leftarrow})\Phi)(x) = \int_X \kappa(\hat{A}; x, x')\Phi(x') \, d\mu(x'), \quad \forall \Phi \in L^2(X), \quad (19)$$

for μ -a.a. $x \in X$, where

$$\kappa(\hat{A}; x, x') := \langle \psi_x, \hat{A}\psi^{x'} \rangle, \quad \forall x, x' \in X; \quad (20)$$

thus $\kappa(x, x') = \kappa(I; x, x')$.

Denoting by $\mathcal{B}_1(\mathcal{S})$ the Banach space of trace class operators in \mathcal{S} , we now prove the following important result:

Proposition 1 (the ‘trace formula for frames’). *With the previous notations and assumptions, for every operator \hat{A} in $\mathcal{B}_1(\mathcal{S})$, the following formula holds:*

$$\text{tr}(\hat{A}) = \int_X \kappa(\hat{A}; x, x) \, d\mu(x). \quad (21)$$

Assume now that the frame $\{\psi_x\}_{x \in X}$ is tight. Then, for every positive bounded operator \hat{B} in \mathcal{S} , $\kappa(\hat{B}; x, x) \geq 0$, and

$$\int_X \kappa(\hat{B}; x, x) \, d\mu(x) < +\infty \quad (22)$$

if and only if \hat{B} is contained in $\mathcal{B}_1(\mathcal{S})$.

Proof. Since, as is well known, every trace class operator \hat{T} admits a decomposition of the form

$$\hat{T} = \hat{T}_1 - \hat{T}_2 + i(\hat{T}_3 - \hat{T}_4), \quad (23)$$

where $\hat{T}_1, \hat{T}_2, \hat{T}_3, \hat{T}_4$ are positive trace class operators, by linearity of the trace we can prove relation (21)—with no loss of generality—for a generic positive trace class operator \hat{A} in \mathcal{S} .

Let us suppose, for the moment, that the frame $\{\psi_x\}_{x \in X}$ is tight; we can assume that it is normalized (i.e. $\mathcal{S}_X = \mathcal{S}^X$). Then, choosing an arbitrary orthonormal basis $\{\eta_n\}_{n \in \mathcal{N}}$ in \mathcal{S} and denoting by $\hat{A}^{\frac{1}{2}}$ the (positive) square root of \hat{A} , we have

$$\begin{aligned} \text{tr}(\hat{A}) &= \sum_{n \in \mathcal{N}} \langle \hat{A}^{\frac{1}{2}}\eta_n, \hat{A}^{\frac{1}{2}}\eta_n \rangle = \sum_{n \in \mathcal{N}} \int_X \langle \hat{A}^{\frac{1}{2}}\eta_n, \psi_x \rangle \langle \psi_x, \hat{A}^{\frac{1}{2}}\eta_n \rangle \, d\mu(x) \\ &= \int_X \sum_{n \in \mathcal{N}} \langle \psi_x, \hat{A}^{\frac{1}{2}}\eta_n \rangle \langle \hat{A}^{\frac{1}{2}}\eta_n, \psi_x \rangle \, d\mu(x) \\ &= \int_X \sum_{n \in \mathcal{N}} \langle \hat{A}^{\frac{1}{2}}\psi_x, \eta_n \rangle \langle \eta_n, \hat{A}^{\frac{1}{2}}\psi_x \rangle \, d\mu(x), \end{aligned} \quad (24)$$

where the permutation of the (possibly infinite) sum with the integral is allowed by the positivity of the integrand functions. Hence, we obtain

$$\text{tr}(\hat{A}) = \int_X \langle \hat{A}^{\frac{1}{2}}\psi_x, \hat{A}^{\frac{1}{2}}\psi_x \rangle \, d\mu(x) = \int_X \langle \psi_x, \hat{A}\psi_x \rangle \, d\mu(x). \quad (25)$$

This proves the first assertion of the statement in the case of a tight frame.

For a generic frame $\{\psi_x\}_{x \in X}$ in \mathcal{S} one can argue as follows. First observe that—denoting, as above, by \hat{M} the metric operator of this frame—the set $\{\tilde{\psi}_x = \hat{M}^{-\frac{1}{2}}\psi_x = \hat{M}^{\frac{1}{2}}\psi^x\}_{x \in X}$ is a

normalized tight frame (exploiting relations (13), the proof of this assertion is straightforward). Next, consider that, for every $\hat{A} \in \mathcal{B}_1(\mathcal{S})$,

$$\begin{aligned} \text{tr}(\hat{A}) &= \text{tr}(\hat{M}^{\frac{1}{2}} \hat{A} \hat{M}^{-\frac{1}{2}}) = \int_X \langle \tilde{\psi}_x, \hat{M}^{\frac{1}{2}} \hat{A} \hat{M}^{-\frac{1}{2}} \tilde{\psi}_x \rangle d\mu(x) \\ &= \int_X \langle \psi_x, \hat{A} \psi^x \rangle d\mu(x), \end{aligned} \tag{26}$$

where we have used the cyclic property of the trace and the result of the first part of the proof.

Let us prove the second assertion of the statement. Assume that the frame $\{\psi_x\}_{x \in X}$ is tight (we can suppose that it is normalized), and let \hat{B} be a positive bounded operator in \mathcal{S} which is *not* contained in $\mathcal{B}_1(\mathcal{S})$. Then, arguing as above, we have

$$\begin{aligned} +\infty &= \sum_{n \in \mathcal{N}} \langle \eta_n, \hat{B} \eta_n \rangle = \sum_{n \in \mathcal{N}} \langle \hat{B}^{\frac{1}{2}} \eta_n, \hat{B}^{\frac{1}{2}} \eta_n \rangle = \int_X \langle \hat{B}^{\frac{1}{2}} \psi_x, \hat{B}^{\frac{1}{2}} \psi_x \rangle d\mu(x) \\ &= \int_X \langle \psi_x, \hat{B} \psi_x \rangle d\mu(x), \end{aligned} \tag{27}$$

where $\{\eta_n\}_{n \in \mathcal{N}}$ is an arbitrary orthonormal basis in \mathcal{S} .

The proof is now complete. □

At this point, we proceed to the second part of the section, where we will specialize the scheme outlined above to the case where $\mathcal{S} = \mathcal{B}_2(\mathcal{H})$, with $\mathcal{B}_2(\mathcal{H})$ denoting the space of Hilbert–Schmidt operators in a (separable complex) Hilbert space \mathcal{H} (we will adopt the symbol $\langle \cdot, \cdot \rangle_{\mathcal{B}_2(\mathcal{H})}$ for denoting the scalar product in $\mathcal{B}_2(\mathcal{H})$: $\langle \hat{A}, \hat{B} \rangle_{\mathcal{B}_2(\mathcal{H})} := \text{tr}(\hat{A}^* \hat{B})$, $\hat{A}, \hat{B} \in \mathcal{B}_2(\mathcal{H})$). We recall the fact that the Hilbert space $\mathcal{B}_2(\mathcal{H})$ is a H^* -algebra [34], and a two-sided $*$ -ideal in the C^* -algebra of bounded operators $\mathcal{B}(\mathcal{H})$ (see, e.g. [35]).

Then, let $\{\hat{T}_y\}_{y \in Y}$ be a frame in $\mathcal{B}_2(\mathcal{H})$, based on a measure space $\mathcal{Y} = (Y, \nu)$, and let $\{\hat{T}^y\}_{y \in Y}$ be the dual frame. In order to avoid confusion, we will now denote by \mathfrak{D} the frame transform associated with the frame $\{\hat{T}_y\}_{y \in Y}$ and by \mathfrak{Q} its pseudo-inverse; thus, we will set

$$\mathfrak{D} \equiv \mathfrak{F} : \mathcal{B}_2(\mathcal{H}) \rightarrow L^2(Y) \equiv L^2(Y, \nu; \mathbb{C}) \quad \text{and} \quad \mathfrak{Q} \equiv \mathfrak{F}^{\leftarrow} : L^2(Y) \rightarrow \mathcal{B}_2(\mathcal{H}). \tag{28}$$

It is natural to wonder if, in addition to the formulae recalled above, one can suitably express the product of operators in $\mathcal{B}_2(\mathcal{H})$ in terms of the frame transforms associated with these operators. Denoting by A, B the frame transforms of $\hat{A}, \hat{B} \in \mathcal{B}_2(\mathcal{H})$, respectively, i.e. $A = \mathfrak{D} \hat{A} := \langle \hat{T}_{(\cdot)}, \hat{A} \rangle_{\mathcal{B}_2(\mathcal{H})} \in L^2(Y)$, $B = \mathfrak{D} \hat{B} \in L^2(Y)$, we can set

$$(A \star B)(y) := (\mathfrak{D} \hat{A} \hat{B})(y). \tag{29}$$

Therefore, the product of operators induces, through the frame transform \mathfrak{D} , a bilinear map $(\cdot) \star (\cdot) : \text{Ran}(\mathfrak{D}) \times \text{Ran}(\mathfrak{D}) \rightarrow \text{Ran}(\mathfrak{D})$. As we are going to show, exploiting the reconstruction formulae

$$\hat{A} = \int_Y A(y) \hat{T}^y d\nu(y), \quad \hat{B} = \int_Y B(y) \hat{T}^y d\nu(y), \tag{30}$$

one can obtain a suitable expression for this bilinear map.

Remark 2. The integrals in the reconstruction formulae (30) are weak integrals of vector-valued functions with respect to the scalar product of $\mathcal{B}_2(\mathcal{H})$. Then, *a fortiori*, they are weak

integrals of bounded-operator-valued functions; indeed

$$\begin{aligned} \left\langle \phi, \left(\int_Y A(y) \hat{T}^y \, d\nu(y) \right) \psi \right\rangle &= \langle |\psi\rangle \langle \phi|, \int_Y A(y) \hat{T}^y \, d\nu(y) \rangle_{\mathcal{B}_2(\mathcal{H})} \\ &= \int_Y A(y) \langle |\psi\rangle \langle \phi|, \hat{T}^y \rangle_{\mathcal{B}_2(\mathcal{H})} \, d\nu(y) \\ &= \int_Y A(y) \langle \phi, \hat{T}^y \psi \rangle \, d\nu(y), \end{aligned} \quad (31)$$

for any couple of vectors $\phi, \psi \in \mathcal{H}$.

It turns out that the bilinear map $(\cdot) \star (\cdot)$, induced through the frame transform by the product of operators in $\mathcal{B}_2(\mathcal{H})$, can be expressed as a ‘non-local’—i.e. non-pointwise—product of functions defined on the range of \mathfrak{D} ; in fact, we have the following result:

Proposition 2. *With the previous notations and assumptions, for any $\hat{A}, \hat{B} \in \mathcal{B}_2(\mathcal{H})$, the following formula holds:*

$$\begin{aligned} (A \star B)(y) &= \int_Y d\nu(y_1) \int_Y d\nu(y_2) \kappa(y, y_1, y_2) A(y_1) B(y_2) \\ &= \int_Y d\nu(y_2) \int_Y d\nu(y_1) \kappa(y, y_1, y_2) A(y_1) B(y_2), \end{aligned} \quad (32)$$

for ν -a.a. $y \in Y$, where the integral kernel $\kappa : Y \times Y \times Y \rightarrow \mathbb{C}$ is defined by

$$\kappa(y, y_1, y_2) := \langle \hat{T}_y, \hat{T}^{y_1} \hat{T}^{y_2} \rangle_{\mathcal{B}_2(\mathcal{H})} = \text{tr}(\hat{T}_y^* \hat{T}^{y_1} \hat{T}^{y_2}). \quad (33)$$

Proof. As anticipated, we will exploit the reconstruction formulae (30). Let us prove the second of relations (32). Observe that, for any $\hat{A}, \hat{B} \in \mathcal{B}_2(\mathcal{H})$, we have

$$(A \star B)(y) := \langle \hat{T}_y, \hat{A} \hat{B} \rangle_{\mathcal{B}_2(\mathcal{H})} = \text{tr}(\hat{T}_y^* \hat{A} \hat{B}) = \text{tr}((\hat{A}^* \hat{T}_y)^* \hat{B}) = \langle \hat{A}^* \hat{T}_y, \hat{B} \rangle_{\mathcal{B}_2(\mathcal{H})}. \quad (34)$$

Hence, using the reconstruction formula for \hat{B} , we find

$$\begin{aligned} (A \star B)(y) &= \langle \hat{A}^* \hat{T}_y, \hat{B} \rangle_{\mathcal{B}_2(\mathcal{H})} = \int_Y d\nu(y_2) \langle \hat{A}^* \hat{T}_y, \hat{T}^{y_2} \rangle_{\mathcal{B}_2(\mathcal{H})} B(y_2) \\ &= \int_Y d\nu(y_2) \langle \hat{T}_y (\hat{T}^{y_2})^*, \hat{A} \rangle_{\mathcal{B}_2(\mathcal{H})} B(y_2), \end{aligned} \quad (35)$$

where we have used the cyclic property of the trace

$$\langle \hat{A}^* \hat{T}_y, \hat{T}^{y_2} \rangle_{\mathcal{B}_2(\mathcal{H})} = \text{tr}(\hat{T}_y^* \hat{A} \hat{T}^{y_2}) = \text{tr}(\hat{T}^{y_2} \hat{T}_y^* \hat{A}) = \langle \hat{T}_y (\hat{T}^{y_2})^*, \hat{A} \rangle_{\mathcal{B}_2(\mathcal{H})}. \quad (36)$$

Next, using the reconstruction formula for \hat{A} , we obtain

$$\begin{aligned} (A \star B)(y) &= \int_Y d\nu(y_2) \langle \hat{T}_y (\hat{T}^{y_2})^*, \hat{A} \rangle_{\mathcal{B}_2(\mathcal{H})} B(y_2) \\ &= \int_Y d\nu(y_2) \int_Y d\nu(y_1) \langle \hat{T}_y (\hat{T}^{y_2})^*, \hat{T}^{y_1} \rangle_{\mathcal{B}_2(\mathcal{H})} A(y_1) B(y_2) \\ &= \int_Y d\nu(y_2) \int_Y d\nu(y_1) \langle \hat{T}_y, \hat{T}^{y_1} \hat{T}^{y_2} \rangle_{\mathcal{B}_2(\mathcal{H})} A(y_1) B(y_2). \end{aligned} \quad (37)$$

The proof the first of relations (32) is similar. □

We will call the non-local product of functions (29) *star product*⁴ associated with the frame $\{\hat{T}_y\}_{y \in Y}$. Let us observe that the definition of the star product of functions in $\text{Ran}(\mathfrak{D})$ can be extended, in a natural way, to all functions in $L^2(Y)$ by setting

$$\Phi_1 \star \Phi_2 := \mathfrak{D}((\mathfrak{Q}\Phi_1)(\mathfrak{Q}\Phi_2)), \quad \forall \Phi_1, \Phi_2 \in L^2(Y). \quad (38)$$

Note that, since $\mathfrak{Q} = \mathfrak{Q}\hat{P}_{\text{Ran}(\mathfrak{D})}$, with $\hat{P}_{\text{Ran}(\mathfrak{D})}$ denoting the orthogonal projection onto $\text{Ran}(\mathfrak{D})$, we have

$$\Phi_1 \star \Phi_2 = (\hat{P}_{\text{Ran}(\mathfrak{D})}\Phi_1) \star (\hat{P}_{\text{Ran}(\mathfrak{D})}\Phi_2). \quad (39)$$

One can easily prove that the ‘extended star product’—namely, the bilinear map $(\cdot) \star (\cdot) : L^2(Y) \times L^2(Y) \rightarrow L^2(Y)$ defined by formula (38)—can be still expressed as a non-local product of functions; indeed:

Proposition 3. *With the previous notations and assumptions, for any $\Phi_1, \Phi_2 \in L^2(Y)$, the following formula holds:*

$$\begin{aligned} (\Phi_1 \star \Phi_2)(y) &= \int_Y dv(y_1) \int_Y dv(y_2) \kappa(y, y_1, y_2) \Phi_1(y_1) \Phi_2(y_2) \\ &= \int_Y dv(y_2) \int_Y dv(y_1) \kappa(y, y_1, y_2) \Phi_1(y_1) \Phi_2(y_2), \end{aligned} \quad (40)$$

for ν -a.a. $y \in Y$.

Proof. Just recall that

$$\mathfrak{Q}\Phi = \int_Y \Phi(y) \hat{T}^y dv(y), \quad \forall \Phi \in L^2(Y), \quad (41)$$

apply definition (38), and argue as in the proof of proposition 2. □

Since $\mathcal{B}_2(\mathcal{H})$ is a two-sided $*$ -ideal in the C^* -algebra $\mathcal{B}(\mathcal{H})$ of bounded operators in \mathcal{H} , for every $\hat{A} \in \mathcal{B}(\mathcal{H})$ one can define the linear maps

$$\mathsf{L}_{\hat{A}} : \mathcal{B}_2(\mathcal{H}) \ni \hat{B} \mapsto \hat{A}\hat{B} \in \mathcal{B}_2(\mathcal{H}) \quad \text{and} \quad \mathsf{R}_{\hat{A}} : \mathcal{B}_2(\mathcal{H}) \ni \hat{B} \mapsto \hat{B}\hat{A} \in \mathcal{B}_2(\mathcal{H}). \quad (42)$$

The maps $\mathsf{L}_{\hat{A}}$ and $\mathsf{R}_{\hat{A}}$ are *bounded* linear operators. Indeed, as is well known [35], we have

$$\|\mathsf{L}_{\hat{A}}\hat{B}\|_{\mathcal{B}_2(\mathcal{H})} \leq \|\hat{A}\| \|\hat{B}\|_{\mathcal{B}_2(\mathcal{H})} \quad \text{and} \quad \|\mathsf{R}_{\hat{A}}\hat{B}\|_{\mathcal{B}_2(\mathcal{H})} \leq \|\hat{A}\| \|\hat{B}\|_{\mathcal{B}_2(\mathcal{H})}; \quad (43)$$

from this relation follows in particular that $\|\mathsf{L}_{\hat{A}}\| \leq \|\hat{A}\|$ and $\|\mathsf{R}_{\hat{A}}\| \leq \|\hat{A}\|$. One can actually show that

$$\|\mathsf{L}_{\hat{A}}\| = \|\mathsf{R}_{\hat{A}}\| = \|\hat{A}\|. \quad (44)$$

Note that, if $\hat{A} \in \mathcal{B}(\mathcal{H})$ is selfadjoint, then the bounded operators $\mathsf{L}_{\hat{A}}$ and $\mathsf{R}_{\hat{A}}$ in $\mathcal{B}_2(\mathcal{H})$ are selfadjoint too. The operators $\mathsf{L}_{\hat{A}}$ and $\mathsf{R}_{\hat{A}}$ are suitably represented in the space of frame transforms $\text{Ran}(\mathfrak{D})$; i.e.

Proposition 4. *For every bounded operator $\hat{A} \in \mathcal{B}(\mathcal{H})$ and every Hilbert–Schmidt operator $\hat{B} \in \mathcal{B}_2(\mathcal{H})$, the following formulae hold:*

$$((\mathfrak{D}\mathsf{L}_{\hat{A}}\mathfrak{Q})B)(y) = (\mathfrak{D}\hat{A}\hat{B})(y) = \int_Y dv(y') \chi^L(\hat{A}; y, y') B(y'), \quad (45)$$

⁴ We recall that the notion of star product of functions on phase space has been extensively studied in the literature; see, e.g., the classical papers [36–38] and the recent contributions [4, 6]. Here we show how a notion of this kind arises in a natural way considering frames of Hilbert–Schmidt operators.

$$((\mathfrak{D}\mathbf{R}_{\hat{A}}\mathfrak{Q})B)(y) = (\mathfrak{D}\hat{B}\hat{A})(y) = \int_Y dv(y')\chi^R(\hat{A}; y, y')B(y'), \quad (46)$$

for *v*-a.a. $y \in Y$, where $B = \mathfrak{D}\hat{B}$ and

$$\chi^L(\hat{A}; y, y') := \langle \hat{T}_y, \hat{A}\hat{T}^{y'} \rangle_{\mathcal{B}_2(\mathcal{H})}, \quad \chi^R(\hat{A}; y, y') := \langle \hat{T}_y, \hat{T}^{y'}\hat{A} \rangle_{\mathcal{B}_2(\mathcal{H})}; \quad (47)$$

hence

$$\hat{A}\hat{B} = \int_Y dv(y_1) \int_Y dv(y_2)\chi^L(\hat{A}; y_1, y_2)B(y_2)\hat{T}^{y_1}, \quad (48)$$

$$\hat{B}\hat{A} = \int_Y dv(y_1) \int_Y dv(y_2)\chi^R(\hat{A}; y_1, y_2)B(y_2)\hat{T}^{y_1}. \quad (49)$$

Moreover, if the frame $\{\hat{T}_y\}_{y \in Y}$ is tight, then, for every operator $\hat{A} \in \mathcal{B}(\mathcal{H})$, we have

$$\chi^L(\hat{A}; y, y') = \chi^L(\hat{A}^*; y', y)^* \quad \text{and} \quad \chi^R(\hat{A}; y, y') = \chi^R(\hat{A}^*; y', y)^*. \quad (50)$$

Proof. Let us prove formula (45). By definition we have

$$(\mathfrak{D}\hat{A}\hat{B})(y) = \langle \hat{T}_y, \hat{A}\hat{B} \rangle_{\mathcal{B}_2(\mathcal{H})}. \quad (51)$$

Then, exploiting the reconstruction formula for the Hilbert–Schmidt operator \hat{B} , we get

$$\begin{aligned} (\mathfrak{D}\hat{A}\hat{B})(y) &= \langle \hat{A}^*\hat{T}_y, \hat{B} \rangle_{\mathcal{B}_2(\mathcal{H})} \\ &= \int_Y dv(y') \langle \hat{A}^*\hat{T}_y, \hat{T}^{y'} \rangle_{\mathcal{B}_2(\mathcal{H})} B(y') \\ &= \int_Y dv(y') \langle \hat{T}_y, \hat{A}\hat{T}^{y'} \rangle_{\mathcal{B}_2(\mathcal{H})} B(y'), \end{aligned} \quad (52)$$

which is what we wanted to prove. The proof of formula (46) is analogous.

Let us suppose, now, that the frame $\{\hat{T}_y\}_{y \in Y}$ is tight. Then, we have

$$\begin{aligned} \chi^L(\hat{A}; y', y)^* &= \text{tr}(\hat{T}_{y'}^* \hat{A} \hat{T}_y) = \text{tr}((\hat{T}_y)^* \hat{A}^* \hat{T}_{y'}) = \text{tr}(\hat{T}_y^* \hat{A}^* \hat{T}_{y'}) \\ &= \chi^L(\hat{A}^*; y, y'). \end{aligned} \quad (53)$$

In a similar way, one proves the analogous relation for the function $\chi^R(\hat{A}; \cdot, \cdot)$.

The proof is complete. \square

It is worth stressing that, for every bounded operator $\hat{A} \in \mathcal{B}(\mathcal{H})$, both the functions $y' \mapsto \chi^L(\hat{A}^*; y', y)$ and $y' \mapsto \chi^R(\hat{A}^*; y', y)$ are contained in $\text{Ran}(\mathfrak{D})$. If the frame $\{\hat{T}_y\}_{y \in Y}$ is tight, due to this fact and to the first of relations (50), for every $\Phi \in L^2(Y)$ we have

$$\begin{aligned} \int_Y dv(y')\chi^L(\hat{A}; y, y')\Phi(y') &= \int_Y dv(y')\chi^L(\hat{A}^*; y', y)^*\Phi(y') \\ &= \int_Y dv(y')\chi^L(\hat{A}^*; y', y)^*(\hat{P}_{\text{Ran}(\mathfrak{D})}\Phi)(y') \\ &= \int_Y dv(y')\chi^L(\hat{A}; y, y')(\hat{P}_{\text{Ran}(\mathfrak{D})}\Phi)(y'). \end{aligned} \quad (54)$$

Assume that the frame $\{\hat{T}_y\}_{y \in Y}$ is tight and normalized (so that \mathfrak{D} is an isometry). Then, since $\hat{P}_{\text{Ran}(\mathfrak{D})} = \mathfrak{D}\mathfrak{Q}$, from the previous relation and from formula (45) we obtain

$$\int_Y dv(y')\chi^L(\hat{A}; y, y')\Phi(y') = (\mathfrak{D}(\hat{A}\mathfrak{Q}\Phi))(y) = (\mathfrak{D}(\hat{A}\mathfrak{Q}\hat{P}_{\text{Ran}(\mathfrak{D})}\Phi))(y), \quad (55)$$

for all $\Phi \in L^2(Y)$; furthermore, for any $\Phi, \Psi \in L^2(Y)$ we have

$$\begin{aligned} \int_Y d\nu(y) \int_Y d\nu(y') \chi^L(\hat{A}; y, y') \Psi(y)^* \Phi(y') &= \langle \hat{P}_{\text{Ran}(\mathfrak{D})} \Psi, \mathfrak{D}(\hat{A} \mathfrak{D} \Phi) \rangle_{\mathcal{B}_2(\mathcal{H})} \\ &= \langle \mathfrak{D} \mathfrak{D} \Psi, \mathfrak{D}(\hat{A} \mathfrak{D} \Phi) \rangle_{\mathcal{B}_2(\mathcal{H})} \\ &= \langle \mathfrak{D} \Psi, \hat{A} \mathfrak{D} \Phi \rangle_{\mathcal{B}_2(\mathcal{H})} \\ &= \langle \mathfrak{D} \hat{P}_{\text{Ran}(\mathfrak{D})} \Psi, \hat{A} \mathfrak{D} \hat{P}_{\text{Ran}(\mathfrak{D})} \Phi \rangle_{\mathcal{B}_2(\mathcal{H})}. \end{aligned} \tag{56}$$

It is obvious that a completely analogous relation holds for the integral kernel $\chi^R(\hat{A}; \cdot, \cdot)$.

Remark 3. Note that the integral kernels $\chi^L(\hat{A}; \cdot, \cdot)$ and $\chi^R(\hat{A}; \cdot, \cdot)$ are nothing but the kernels of the bounded (super-)operators $L_{\hat{A}}$ and $R_{\hat{A}}$ with respect to the frame $\{\hat{T}_y\}_{y \in Y}$ (see formula (20)). The ‘left’ and ‘right’ integral kernels form vector spaces that can be endowed with the structure of a C^* -algebra isomorphic to the algebra of bounded operators $\mathcal{B}(\mathcal{H})$. Differently from the case of $\text{Ran}(\mathfrak{D}) \equiv \text{Ran}(\mathfrak{F})$, we will assume that these vector spaces are composed of functions rather than of equivalence classes of functions (see remark 1). Observe, moreover, that for any $\hat{A}_1, \hat{A}_2 \in \mathcal{B}(\mathcal{H})$ we have

$$\chi^L(\hat{A}_1 \hat{A}_2; y_1, y_2) = \int_Y d\nu(y) \chi^L(\hat{A}_1; y_1, y) \chi^L(\hat{A}_2; y, y_2), \tag{57}$$

for all $y_1 \in Y$ and all $y_2 \in Y$; indeed, exploiting the resolution of the identity generated by the frame $\{\hat{T}_y\}_{y \in Y}$, we get

$$\begin{aligned} \chi^L(\hat{A}_1 \hat{A}_2; y_1, y_2) &= \langle \hat{T}_{y_1}, \hat{A}_1 \hat{A}_2 \hat{T}^{y_2} \rangle_{\mathcal{B}_2(\mathcal{H})} = \langle \hat{A}_1^* \hat{T}_{y_1}, \hat{A}_2 \hat{T}^{y_2} \rangle_{\mathcal{B}_2(\mathcal{H})} \\ &= \int_Y d\nu(y) \langle \hat{A}_1^* \hat{T}_{y_1}, \hat{T}^y \rangle_{\mathcal{B}_2(\mathcal{H})} \langle \hat{T}_y, \hat{A}_2 \hat{T}^{y_2} \rangle_{\mathcal{B}_2(\mathcal{H})} \\ &= \int_Y d\nu(y) \chi^L(\hat{A}_1; y_1, y) \chi^L(\hat{A}_2; y, y_2). \end{aligned} \tag{58}$$

Clearly, an analogous expression holds for the integral kernel $\chi^R(\hat{A}_1 \hat{A}_2; \cdot, \cdot)$, i.e.

$$\chi^R(\hat{A}_1 \hat{A}_2; y_1, y_2) = \int_Y d\nu(y) \chi^R(\hat{A}_2; y_1, y) \chi^R(\hat{A}_1; y, y_2). \tag{59}$$

Therefore—denoting by $\mathcal{B}(\mathcal{H})_{\mathbb{R}}$ the Jordan–Lie algebra [39] of bounded selfadjoint operators in \mathcal{H} , endowed with the *Jordan product* $\hat{A}_1 \circ \hat{A}_2 := \frac{1}{2}(\hat{A}_1 \hat{A}_2 + \hat{A}_2 \hat{A}_1)$ and with the *Lie bracket* $\{\hat{A}_1, \hat{A}_2\} := \frac{1}{i}[\hat{A}_1, \hat{A}_2]$ —we find

$$\begin{aligned} \chi^L(\hat{A}_1 \circ \hat{A}_2; y_1, y_2) &= \frac{1}{2} \int_Y d\nu(y) (\chi^L(\hat{A}_1; y_1, y) \chi^L(\hat{A}_2; y, y_2) \\ &\quad + \chi^L(\hat{A}_2; y_1, y) \chi^L(\hat{A}_1; y, y_2)), \end{aligned} \tag{60}$$

$$\begin{aligned} \chi^L(\{\hat{A}_1, \hat{A}_2\}; y_1, y_2) &= \frac{1}{i} \int_Y d\nu(y) (\chi^L(\hat{A}_1; y_1, y) \chi^L(\hat{A}_2; y, y_2) \\ &\quad - \chi^L(\hat{A}_2; y_1, y) \chi^L(\hat{A}_1; y, y_2)), \end{aligned} \tag{61}$$

for any $\hat{A}_1, \hat{A}_2 \in \mathcal{B}(\mathcal{H})_{\mathbb{R}}$. Analogous relations hold for $\chi^R(\hat{A}_1 \circ \hat{A}_2; \cdot, \cdot)$ and $\chi^R(\{\hat{A}_1, \hat{A}_2\}; \cdot, \cdot)$.

Let us now suppose to have, simultaneously, a *couple* of frames: the frame $\{\hat{T}_y\}_{y \in Y}$ in the space of Hilbert–Schmidt operators $\mathcal{B}_2(\mathcal{H})$ and a frame $\{\psi_x\}_{x \in X}$ in the Hilbert space \mathcal{H} , based on a measure space $\mathcal{X} = (X, \mu)$. A situation of this kind will be considered in section 7. Then, in addition to the collection of formulae previously obtained, we have the following result:

Proposition 5. For every bounded operator $\hat{A} \in \mathcal{B}(\mathcal{H})$, every Hilbert–Schmidt operator $\hat{B} \in \mathcal{B}_2(\mathcal{H})$ and every trace class operator $\hat{\rho} \in \mathcal{B}_1(\mathcal{H})$, the following formulae hold:

$$\kappa(\hat{B}; x, x') := \langle \psi_x, \hat{B}\psi^{x'} \rangle = \int_Y dv(y)\Gamma(x, x', y)B(y), \tag{62}$$

$$\text{tr}(\hat{\rho}) = \int_X d\mu(x) \int_Y dv(y)\gamma(x, y)\rho(y), \tag{63}$$

$$\begin{aligned} \text{tr}(\hat{A}\hat{\rho}) &= \int_X d\mu(x) \int_Y dv(y_1) \int_Y dv(y_2)\gamma(x, y_1)\chi^{\perp}(\hat{A}; y_1, y_2)\rho(y_2) \\ &= \int_X d\mu(x) \int_Y dv(y_1) \int_Y dv(y_2)\gamma(x, y_1)\chi^{\text{R}}(\hat{A}; y_1, y_2)\rho(y_2), \end{aligned} \tag{64}$$

where $B = \mathfrak{D}\hat{B}$, $\rho = \mathfrak{D}\hat{\rho}$, and we have set: $\Gamma(x, x', y) := \langle \psi_x, \hat{T}^y\psi^{x'} \rangle$, $\gamma(x, y) := \langle \psi_x, \hat{T}^y\psi^x \rangle = \Gamma(x, x, y)$. Assume now that the frame $\{\psi_x\}_{x \in X}$ is tight. Then, for every positive Hilbert–Schmidt operator \hat{B} in \mathcal{H} , $\int_Y dv(y)\gamma(x, y)(\mathfrak{D}\hat{B})(y) = \langle \psi_x, \hat{B}\psi^x \rangle \propto \langle \psi_x, \hat{B}\psi_x \rangle \geq 0$, and

$$\int_X d\mu(x) \int_Y dv(y)\gamma(x, y)(\mathfrak{D}\hat{B})(y) < +\infty \tag{65}$$

if and only if \hat{B} is contained in $\mathcal{B}_1(\mathcal{H})$.

Proof. Taking into account remark 2, formula (62) follows from the reconstruction formula for the operator \hat{B} .

Let us prove formula (63). Applying the trace formula (21) to $\hat{\rho}$, and using formula (62) for the integral kernel $\kappa(\hat{\rho}; \cdot, \cdot)$, we get

$$\text{tr}(\hat{\rho}) = \int_X d\mu(x)\kappa(\hat{\rho}; x, x) = \int_X d\mu(x) \int_Y dv(y)\Gamma(x, x, y)\rho(y). \tag{66}$$

Let us now prove the first of relations (64). Applying formula (63) to the trace class operator $\hat{A}\hat{\rho}$, we get

$$\text{tr}(\hat{A}\hat{\rho}) = \int_X d\mu(x) \int_Y dv(y_1)\gamma(x, y_1)(\mathfrak{D}\hat{A}\hat{\rho})(y_1). \tag{67}$$

Next, by virtue of formula (45), we obtain

$$\text{tr}(\hat{A}\hat{\rho}) = \int_X d\mu(x) \int_Y dv(y_1)\gamma(x, y_1) \int_Y dv(y_2)\chi^{\perp}(\hat{A}; y_1, y_2)\rho(y_2), \tag{68}$$

where $\rho = \mathfrak{D}\hat{\rho}$. The proof of the second of relations (64) is analogous.

The proof of the second assertion of the statement follows from the second assertion of proposition 1. \square

The frame transform $\mathfrak{D} \equiv \mathfrak{F}$ associated with a frame in $\mathcal{B}_2(\mathcal{H})$ may be regarded as a ‘dequantization map’, which associates with any operator in $\mathcal{B}_2(\mathcal{H})$ a square integrable function. Conversely, the pseudo-inverse $\mathfrak{Q} \equiv \mathfrak{F}^{\leftarrow}$ may be regarded as a ‘quantization map’ which suitably associates an operator with a \mathbb{C} -valued function. In this context, the counterpart of the product of operators is given by the star product of functions. At this point, the reader must have recognized the typical scheme underlying the subject which is usually called ‘quantum mechanics on phase space’: the Wigner transform (dequantization), the Weyl map (quantization) and the Grönewold–Moyal product of functions (star product), see [3]. In the following, we will show that there is a precise link between the ‘frame formalism’ discussed in the present section and the Weyl–Wigner–Grönewold–Moyal formalism for quantum mechanics.

3. Quantum mechanics on phase space: the Wigner distribution

As is well known, due to the indetermination relations, the notion of phase space is not straightforward in the quantum-mechanical setting as is in the classical setting. Since particles cannot have, simultaneously, a well-defined position q and momentum p , it is not possible to define a genuine phase space probability distribution for a quantum particle as it happens in classical statistical mechanics; in other words, quantum mechanics is not a statistical theory in the classical sense. It is, however, possible to introduce a notion of ‘quasi-probability distribution’ or ‘quasi-distribution’ that allows one to express quantum averages in a way analogous to classical averages.

In the following, for the sake of notational simplicity, we will consider the case of a $(1 + 1)$ -dimensional phase space (with coordinates denoted by q, p); the extension to the ordinary $(3 + 3)$ -dimensional case is straightforward. In the classical setting, a particle can be described by a classical probability distribution on phase space $(q, p) \mapsto \mathcal{P}(q, p)$ (or, more generally, by a probability measure). The average (at a certain time) of a function of position and momentum $(q, p) \mapsto A(q, p)$ —namely, of a classical observable—is given by the expression

$$\langle A \rangle_{\mathcal{P}} = \int_{\mathbb{R} \times \mathbb{R}} A(q, p) \mathcal{P}(q, p) \, dq \, dp. \quad (69)$$

On the other hand, a quantum-mechanical state is described by a density operator $\hat{\rho}$ —a positive trace class operator of unit trace—and the mean value of a quantum observable \hat{A} , which (by virtue of the spectral decomposition of a selfadjoint operator) can always be assumed to be a *bounded* selfadjoint operator, is given by the well-known ‘trace formula’

$$\langle \hat{A} \rangle_{\hat{\rho}} = \text{tr}(\hat{A} \hat{\rho}). \quad (70)$$

If one tries to establish a link between the classical formula (69) and the quantum one (70), one has to face the following problem: how one can *set a suitable correspondence between a quantum observable \hat{A} (i.e. a selfadjoint operator, in the standard formulation of quantum mechanics) and a ‘corresponding classical-like observable’ $(q, p) \mapsto A(q, p)$ (a numerical function), and between a density operator $\hat{\rho}$ and a suitable ‘quantum quasi-distribution function’ $(q, p) \mapsto \mathcal{Q}_{\hat{\rho}}(q, p)$, in such a way that it is then possible to express the expectation value of a quantum observable in a ‘formally classical fashion’, i.e. as a phase space average of the type (69)*

$$\langle \hat{A} \rangle_{\hat{\rho}} = \int_{\mathbb{R} \times \mathbb{R}} A(q, p) \mathcal{Q}_{\hat{\rho}}(q, p) \, dq \, dp. \quad (71)$$

It is a remarkable fact that this problem can be solved—at least partially—within a theoretical scheme usually called ‘Weyl–Wigner formulation of quantum mechanics’, or, in a slightly more general sense, ‘phase space formulation of quantum mechanics’. It turns out that the correspondence *operator* \leftrightarrow *numerical function* is of the same kind (i.e. it is obtained using the same formulae) both for the density operator $\hat{\rho}$ and the observable \hat{A} (at least for a suitable class of observables).

As is well known, the notion of quasi-distribution function has been introduced by Wigner in his celebrated paper [1], with the aim of exploring the quantum corrections to classical statistical mechanics. The quasi-distribution introduced by Wigner—which is still regarded nowadays as the ‘standard’ quasi-distribution function (other quasi-distributions, with remarkable applications in quantum optics, can also be defined, see [18, 40, 41]; see also the recent proposals [42, 43])—is universally known as the *Wigner distribution*. In the following, we will recall a few basic results; for the proofs, the reader may consult standard

references on the subject like [29, 30]. As above, in order to simplify notation, we will consider the case of a quantum particle with a single degree of freedom (hence, we will deal with a (1 + 1)-dimensional phase space). Then, let us denote by ψ a vector in the Hilbert space $L^2(\mathbb{R})$ and, using the Dirac notation, let us set $\hat{\psi} \equiv |\psi\rangle\langle\psi|$. With the vector ψ —or, more precisely, with the operator $\hat{\psi}$ —one can associate the function

$$\mathcal{Q}_{\hat{\psi}} : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{C}, \tag{72}$$

defined by ($\hbar = 1$)

$$\mathcal{Q}_{\hat{\psi}}(q, p) := \frac{1}{2\pi} \int_{\mathbb{R}} e^{-ipx} \psi\left(q - \frac{x}{2}\right)^* \psi\left(q + \frac{x}{2}\right) dx. \tag{73}$$

If $\psi \in L^2(\mathbb{R})$ is, in particular, a normalized nonzero vector (i.e. $\|\psi\| = 1$), then $\mathcal{Q}_{\hat{\psi}}$ is called the ‘Wigner distribution associated with the pure state $\hat{\psi}$ ’. Note that, for almost all $q \in \mathbb{R}$, the function

$$\mathbb{R} \ni x \mapsto \psi\left(q - \frac{x}{2}\right)^* \psi\left(q + \frac{x}{2}\right) \in \mathbb{C} \tag{74}$$

is contained in $L^1(\mathbb{R})$; hence the Fourier integral in definition (73) is indeed an ordinary integral. Moreover, this integral can be regarded as $1/\pi$ times the scalar product of the normalized functions

$$\mathbb{R} \ni x \mapsto \frac{e^{ipx}}{\sqrt{2}} \psi\left(q - \frac{x}{2}\right) \in \mathbb{C} \quad \text{and} \quad \mathbb{R} \ni x \mapsto \frac{1}{\sqrt{2}} \psi\left(q + \frac{x}{2}\right) \in \mathbb{C}; \tag{75}$$

hence, according to the Cauchy–Schwarz inequality, we have

$$|\mathcal{Q}_{\hat{\psi}}(q, p)| \leq \frac{1}{\pi} \|\psi\|^2, \quad \forall \psi \in L^2(\mathbb{R}), \quad \forall q, p \in \mathbb{R}. \tag{76}$$

One can easily prove that the function $\mathcal{Q}_{\hat{\psi}}$ assumes only real values.

As far as we know, it is not completely clear in what way Wigner obtained formula (73). It seems that he achieved this expression by requiring that some general properties were satisfied in a ‘simple way’ (see [44] and references therein); in particular:

- (i) As already mentioned, the function $\mathcal{Q}_{\hat{\psi}}$ assumes only real values.
- (ii) The *marginal sub-distributions*

$$\mathcal{Q}_{\hat{\psi}}(q, \cdot) : \mathbb{R} \ni p \mapsto \mathcal{Q}_{\hat{\psi}}(q, p), \quad q \in \mathbb{R}, \quad \mathcal{Q}_{\hat{\psi}}(\cdot, p) : \mathbb{R} \ni q \mapsto \mathcal{Q}_{\hat{\psi}}(q, p), \quad p \in \mathbb{R}, \tag{77}$$

satisfy the following relations:

$$\int_{\mathbb{R}} \mathcal{Q}_{\hat{\psi}}(q, p) dp = |\psi(q)|^2, \quad \text{for a.a. } q \in \mathbb{R}, \tag{78}$$

$$\int_{\mathbb{R}} \mathcal{Q}_{\hat{\psi}}(q, p) dq = |(\mathcal{F}\psi)(p)|^2, \quad \text{for a.a. } p \in \mathbb{R}, \tag{79}$$

where $\mathcal{F} : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ is the Fourier–Plancherel operator. We remark that, rigorously, the function $\mathcal{Q}_{\hat{\psi}}$ and the associated marginal sub-distributions are not integrable, in general. However, one can easily prove that, if $\mathcal{F}\psi$ belongs to $L^1(\mathbb{R})$ (hence, $\mathcal{F}\psi \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$), then the marginal sub-distribution $\mathcal{Q}_{\hat{\psi}}(q, \cdot)$ is contained in $L^1(\mathbb{R})$ too and relation (78) holds true. Analogously, if ψ belongs to $L^1(\mathbb{R})$ ($\cap L^2(\mathbb{R})$), then $\mathcal{Q}_{\hat{\psi}}(\cdot, p)$ is contained in $L^1(\mathbb{R})$ and relation (79) is satisfied as well. Note that, if relation (78) holds (in particular, if $\mathcal{F}\psi \in L^1(\mathbb{R})$), then

$$\int_{\mathbb{R}} \left(\int_{\mathbb{R}} \mathcal{Q}_{\hat{\psi}}(q, p) dp \right) dq = \|\psi\|^2; \tag{80}$$

similarly, if relation (79) holds (in particular, if $\psi \in L^1(\mathbb{R})$), then

$$\int_{\mathbb{R}} \left(\int_{\mathbb{R}} Q_{\hat{\psi}}(q, p) dq \right) dp = \|\psi\|^2. \tag{81}$$

Moreover, it is possible to prove that if ψ belongs to the Schwartz space $\mathcal{S}(\mathbb{R})$, then $Q_{\hat{\psi}}$ belongs to $\mathcal{S}(\mathbb{R} \times \mathbb{R})$; thus, both relations (78) and (79) hold true, and we have that

$$\int_{\mathbb{R} \times \mathbb{R}} Q_{\hat{\psi}}(q, p) dq dp = \|\psi\|^2. \tag{82}$$

However, we stress that, for $\|\psi\| = 1$, the Wigner distribution associated with the pure state $\hat{\psi}$ cannot be regarded as a genuine probability distribution as it assumes, in general, both positive and negative values (this fact is already explicitly observed in Wigner’s original paper [1]).

- (iii) The function $Q_{\hat{\psi}}$ behaves in an ‘elementary way’ with respect to position and momentum translations, namely

$$\psi(q) \mapsto \psi(q - q') = (e^{-iq'\hat{p}}\psi)(q) \implies Q_{\hat{\psi}}(q, p) \mapsto Q_{\hat{\psi}}(q - q', p), \tag{83}$$

$$\psi(q) \mapsto e^{ip'q}\psi(q) = (e^{ip'\hat{q}}\psi)(q) \implies Q_{\hat{\psi}}(q, p) \mapsto Q_{\hat{\psi}}(q, p - p'), \tag{84}$$

where we have denoted by \hat{q} and \hat{p} the standard position and momentum operators in $L^2(\mathbb{R})$, respectively.

However, we point out that it is the peculiar property of satisfying a relation of the type (71) for the expectation values of observables, the salient feature of the Wigner distribution. As will be shown later on, one can actually associate with *any* trace class operator in $L^2(\mathbb{R})$ (in particular, with any physical state, i.e. not only with a pure state) a suitable (generalized) Wigner distribution; this association will then allow us to obtain an expression of the type (71). The first step of this generalization is to associate with any finite-rank operator a Wigner distribution (we will not attempt at establishing formula (71) itself, for the moment). To this aim, for any couple of vectors ϕ, ψ in $L^2(\mathbb{R})$, let us set

$$Q_{\widehat{\phi\psi}}(q, p) := \frac{1}{2\pi} \int_{\mathbb{R}} e^{-ipx} \psi \left(q - \frac{x}{2} \right)^* \phi \left(q + \frac{x}{2} \right) dx; \tag{85}$$

this expression is a straightforward generalization of formula (73), relating a *generic* rank-one operator $\widehat{\phi\psi} \equiv |\phi\rangle\langle\psi|$ with a \mathbb{C} -valued function. Note that, as in the case of $Q_{\hat{\psi}} \equiv Q_{\widehat{\psi\psi}}$, the function $Q_{\widehat{\phi\psi}}$ is well defined since that map $x \mapsto \phi \left(q - \frac{x}{2} \right)^* \psi \left(q + \frac{x}{2} \right)$ belongs to $L^2(\mathbb{R})$ for all $q \in \mathbb{R}$. It is also immediate to observe that, for any $q, p \in \mathbb{R}$, $|Q_{\widehat{\phi\psi}}(q, p)| \leq \frac{1}{\pi} \|\phi\| \|\psi\|$, and

$$\begin{aligned} Q_{\widehat{\phi\psi}}(q, p)^* &= \frac{1}{2\pi} \int_{\mathbb{R}} e^{ipx} \psi \left(q - \frac{x}{2} \right) \phi \left(q + \frac{x}{2} \right)^* dx \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} e^{-ipx} \phi \left(q - \frac{x}{2} \right)^* \psi \left(q + \frac{x}{2} \right) dx, \end{aligned} \tag{86}$$

hence $Q_{\widehat{\phi\psi}}(q, p)^* = Q_{\widehat{\psi\phi}}(q, p), \forall \phi, \psi \in L^2(\mathbb{R})$. One can prove, moreover, that for any $\phi, \psi \in L^2(\mathbb{R})$ the function $Q_{\widehat{\phi\psi}}$ is contained in $L^2(\mathbb{R} \times \mathbb{R})$, and the following important relation—the *Moyal identity*—holds

$$\int_{\mathbb{R} \times \mathbb{R}} Q_{\widehat{\phi_1\psi_1}}(q, p)^* Q_{\widehat{\phi_2\psi_2}}(q, p) dq dp = \frac{1}{2\pi} \langle \phi_1, \phi_2 \rangle \langle \psi_2, \psi_1 \rangle = \frac{1}{2\pi} \text{tr}(\widehat{\phi_1\psi_1}^* \widehat{\phi_2\psi_2}), \tag{87}$$

for all $\phi_1, \psi_1, \phi_2, \psi_2 \in L^2(\mathbb{R})$; in particular, for $\phi_1 = \psi_1 = \phi_2 = \psi_2 \equiv \psi$, and recalling that $\mathcal{Q}_{\widehat{\psi}}(q, p) \in \mathbb{R}$, we have

$$\int_{\mathbb{R} \times \mathbb{R}} \mathcal{Q}_{\widehat{\psi}}(q, p)^2 dq dp = \frac{1}{2\pi} \|\psi\|^4 \tag{88}$$

(compare with formula (82); note, however, that formula (88) holds for *every* vector ψ in $L^2(\mathbb{R})$).

Consider now the family of unitary operators $\{U(q, p)\}_{q, p \in \mathbb{R}} \subset \mathcal{U}(L^2(\mathbb{R}))$ (given a Hilbert space \mathcal{H} , we denote by $\mathcal{U}(\mathcal{H})$ the unitary group of \mathcal{H}) defined by

$$U(q, p) := \exp(i(p\hat{q} - q\hat{p})) \\ = e^{-\frac{i}{2}qp} \exp(ip\hat{q}) \exp(-iq\hat{p}) = e^{\frac{i}{2}qp} \exp(-iq\hat{p}) \exp(ip\hat{q}), \quad q, p \in \mathbb{R}. \tag{89}$$

One can prove (see [12]) that the function $\text{tr}(U(\cdot, \cdot)^* \widehat{\phi\psi}) : (q, p) \mapsto \text{tr}(U(q, p)^* \widehat{\phi\psi})$ belongs to $L^2(\mathbb{R} \times \mathbb{R})$ and the following relation holds:

$$\mathcal{Q}_{\widehat{\phi\psi}}(q, p) = \frac{1}{2\pi} (\mathcal{F}_{sp} \text{tr}(U(\cdot, \cdot)^* \widehat{\phi\psi}))(q, p), \tag{90}$$

where $\mathcal{F}_{sp} : L^2(\mathbb{R} \times \mathbb{R}) \rightarrow L^2(\mathbb{R} \times \mathbb{R})$ is the *symplectic Fourier transform*, i.e. the unitary operator determined by

$$(\mathcal{F}_{sp} f)(q, p) = \frac{1}{2\pi} \int_{\mathbb{R} \times \mathbb{R}} f(q', p') e^{i(qp' - p'q')} dq' dp', \quad \forall f \in L^1(\mathbb{R} \times \mathbb{R}) \cap L^2(\mathbb{R} \times \mathbb{R}). \tag{91}$$

Recall that \mathcal{F}_{sp} enjoys the remarkable property of being both unitary and selfadjoint

$$\mathcal{F}_{sp} = \mathcal{F}_{sp}^*, \quad \mathcal{F}_{sp}^2 = I. \tag{92}$$

Thus, for any $\phi, \psi \in L^2(\mathbb{R})$, the Wigner distribution is the symplectic Fourier transform of the function

$$\mathcal{V}_{\widehat{\phi\psi}} : \mathbb{R} \times \mathbb{R} \ni (q, p) \mapsto (2\pi)^{-1} \text{tr}(U(q, p)^* \widehat{\phi\psi}) \in \mathbb{C}, \tag{93}$$

which is usually called *Fourier–Wigner distribution* associated with the rank-one operator $\widehat{\phi\psi}$. It is a peculiar fact that the Fourier–Wigner distribution can be cast in a form similar to the standard Wigner distribution (compare with formula (85))

$$\mathcal{V}_{\widehat{\phi\psi}}(q, p) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{\frac{i}{2}qp} e^{-ipy} \psi(y - q)^* \phi(y) dy \\ = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-ipx} \psi\left(x - \frac{q}{2}\right)^* \phi\left(x + \frac{q}{2}\right) dx. \tag{94}$$

It is clear that, since \mathcal{F}_{sp} is unitary, the function $\mathcal{V}_{\widehat{\phi\psi}} = \mathcal{F}_{sp} \mathcal{Q}_{\widehat{\phi\psi}}$ satisfies a relation completely analogous to the Moyal identity (87).

As is well known, the map $\mathbb{R} \times \mathbb{R} \ni (q, p) \mapsto U(q, p)$ that appears in the definition of the Wigner and Fourier–Wigner distributions is an *irreducible projective representation* of the group $\mathbb{R} \times \mathbb{R}$ in $L^2(\mathbb{R})$; with a slight abuse of terminology, we will call it *Weyl system*⁵. The Moyal identity (87) is a manifestation of the fact that the representation U is *square integrable*. This property, whose main technical aspects will be recalled in the following section, allows us to extend the notion of Wigner distribution defining a *Wigner transform* which associates with any Hilbert–Schmidt operator in $L^2(\mathbb{R})$ a suitable numerical function; furthermore, as will be shown in section 5, one can actually define a (generalized) Wigner transform for *every* square integrable representation.

⁵ Strictly speaking, it is the pair of unitary representations ($p \mapsto \exp(ip\hat{q}), q \mapsto \exp(-iq\hat{p})$) that it is customary to call ‘Weyl system’, see [45]; however, the irreducible projective representation U has the same physical meaning since it ‘codifies’ the canonical commutation relations (in integrated form), as shown in (89). The representation U is strictly related to a *Schrödinger representation* of the Heisenberg–Weyl group, see [29].

4. A technical interlude: square integrable representations

In this section, we will use some basic facts of the theory of topological groups and their representations; standard references on the subject are [46, 47].

Let G be a locally compact second countable Hausdorff topological group (in short, l.c.s.c. group). We will denote by μ_G and Δ_G respectively a *left Haar measure* (of course uniquely defined up to a multiplicative constant) and the *modular function* on G . The symbol e will indicate the unit element in G .

Given a separable complex Hilbert space \mathcal{H} , the symbol $\mathcal{U}(\mathcal{H})$ will denote, as in section 3, the *unitary group* of \mathcal{H} —i.e. the group of all unitary operators in \mathcal{H} , endowed with the strong operator topology—which is a metrizable second countable Hausdorff topological group.

We will mean by the term *projective representation* of a l.c.s.c. group G a Borel projective representation of G in a separable complex Hilbert space \mathcal{H} (see, for instance, [46], chapter VII), namely a map of G into $\mathcal{U}(\mathcal{H})$ such that

- U is a weakly Borel map, i.e. $G \ni g \mapsto \langle \phi, U(g)\psi \rangle \in \mathbb{C}$ is a Borel function⁶, for any couple of vectors $\phi, \psi \in \mathcal{H}$;
- $U(e) = I$, where I the identity operator in \mathcal{H} ;
- denoted by \mathbb{T} the circle group, namely the group of complex numbers of modulus one, there exists a Borel function $m : G \times G \rightarrow \mathbb{T}$ such that

$$U(gh) = m(g, h)U(g)U(h), \quad \forall g, h \in G.$$

The function m , which is called the *multiplier associated with U* , satisfies the following conditions:

$$m(g, e) = m(e, g) = 1, \quad \forall g \in G, \tag{95}$$

and

$$m(g_1, g_2g_3)m(g_2, g_3) = m(g_1g_2, g_3)m(g_1, g_2), \quad \forall g_1, g_2, g_3 \in G. \tag{96}$$

In particular, in the case where $m \equiv 1$, U is a standard unitary representation; in this case, according to a well-known result, the hypothesis that the map U is weakly Borel implies that it is, actually, strongly continuous. The notion of irreducibility is defined for projective representations as for unitary representations.

Let $\tilde{U} : G \rightarrow \mathcal{U}(\tilde{\mathcal{H}})$ be a projective representation of G in a (separable complex) Hilbert space $\tilde{\mathcal{H}}$. We say that \tilde{U} is *physically equivalent* to U if there exist a Borel function $\beta : G \rightarrow \mathbb{T}$ and a unitary or antiunitary operator $W : \mathcal{H} \rightarrow \tilde{\mathcal{H}}$ such that

$$\tilde{U}(g) = \beta(g)WU(g)W^*, \quad \forall g \in G. \tag{97}$$

Note that the notion of physical equivalence is coherent with Wigner’s theorem on symmetry actions. It is clear that a projective representation, physically equivalent to an irreducible projective representation, is irreducible too.

Let U be an *irreducible* projective representation of the l.c.s.c. group G in the Hilbert space \mathcal{H} . Then, given two vectors $\psi, \phi \in \mathcal{H}$, we define the function (usually called ‘coefficient’)

$$c_{\psi, \phi}^U : G \ni g \mapsto \langle U(g)\psi, \phi \rangle \in \mathbb{C}, \tag{98}$$

and we consider the set (of ‘admissible vectors for U ’)

$$\mathcal{A}(U) := \{ \psi \in \mathcal{H} \mid \exists \phi \in \mathcal{H} : \phi \neq 0, c_{\psi, \phi}^U \in L^2(G) \}, \tag{99}$$

⁶ The terms *Borel function* (or map) and *Borel measure* will be always used with reference to the natural Borel structures on the topological spaces involved, namely to the smallest σ -algebras containing all open subsets.

where $L^2(G) \equiv L^2(G, \mu_G; \mathbb{C})$. The representation U is said to be *square integrable* if $\mathcal{A}(U) \neq \{0\}$. Square integrable projective representations are characterized by the following result—see [22]—which is a generalization of a classical theorem of Duflo and Moore [19] concerning unitary representations:

Theorem 1. *Let the projective representation $U : G \rightarrow \mathcal{U}(\mathcal{H})$ be square integrable. Then, the set $\mathcal{A}(U)$ is a dense linear span⁷ in \mathcal{H} , stable under the action of U , and, for any couple of vectors $\phi \in \mathcal{H}$ and $\psi \in \mathcal{A}(U)$, the coefficient $c_{\psi, \phi}^U$ is square integrable with respect to the left Haar measure μ_G on G . Moreover, there exists a unique positive selfadjoint injective linear operator \hat{D}_U in \mathcal{H} —which we will call the ‘Duflo–Moore operator associated with U ’—such that $\mathcal{A}(U) = \text{Dom}(\hat{D}_U)$ and the following ‘orthogonality relations’ hold:*

$$\begin{aligned} \int_G c_{\psi_1, \phi_1}(g)^* c_{\psi_2, \phi_2}(g) d\mu_G(g) &= \int_G \langle \phi_1, U(g)\psi_1 \rangle \langle U(g)\psi_2, \phi_2 \rangle d\mu_G(g) \\ &= \langle \phi_1, \phi_2 \rangle \langle \hat{D}_U \psi_2, \hat{D}_U \psi_1 \rangle, \end{aligned} \tag{100}$$

for all $\phi_1, \phi_2 \in \mathcal{H}$ and all $\psi_1, \psi_2 \in \mathcal{A}(U)$. The Duflo–Moore operator \hat{D}_U is semi-invariant, with respect to U , with weight $\Delta_G^{1/2}$, i.e.

$$U(g)\hat{D}_U = \Delta_G(g)^{\frac{1}{2}} \hat{D}_U U(g), \quad \forall g \in G; \tag{101}$$

it is bounded if and only if G is unimodular (i.e. $\Delta_G \equiv 1$) and, in such a case, it is a multiple of the identity.

Remark 4. If U is square integrable, the associated Duflo–Moore operator \hat{D}_U , being injective and selfadjoint, has a selfadjoint densely defined inverse. Duflo and Moore call (for historical reasons) the square of \hat{D}_U^{-1} the *formal degree* of the representation U . Note that the operator \hat{D}_U is linked to the normalization of the Haar measure μ_G . Indeed, if μ_G is rescaled by a positive constant, then \hat{D}_U is rescaled by the square root of the same constant. Keeping this fact in mind, we will say that \hat{D}_U is *normalized according to μ_G* . On the other hand, if a normalization of the left Haar measure on G is not fixed, \hat{D}_U is defined up to a positive factor and we will call a specific choice a *normalization of the Duflo–Moore operator*. In particular, if G is unimodular, then $\hat{D}_U = I$ is a normalization of the Duflo–Moore operator; the corresponding Haar measure will be said to be *normalized in agreement with the representation U* . Moreover, observe that, according to relation (101), the dense linear span $\text{Dom}(\hat{D}_U^{-1}) = \text{Ran}(\hat{D}_U)$ (like the linear span $\mathcal{A}(U) = \text{Dom}(\hat{D}_U)$) is stable under the action of U and

$$U(g)^* \hat{D}_U^{-1} = \Delta_G(g)^{\frac{1}{2}} \hat{D}_U^{-1} U(g)^*, \quad \forall g \in G. \tag{102}$$

Finally, we note that the orthogonality relations (100) can also be written replacing the positive selfadjoint operator \hat{D}_U with a closed injective operator \hat{K}_U which is only required to be selfadjoint. Such an operator \hat{K}_U is not unique (e.g., trivially, one can set $\hat{K}_U = -\hat{D}_U$), and it is characterized by a polar decomposition of the form $\hat{K}_U = V \hat{D}_U$, where V is a suitable unitary operator in \mathcal{H} . A selfadjoint operator satisfying the orthogonality relations will be called a *variant of the Duflo–Moore operator*.

Let us list a few basic facts about square integrable representations:

- (i) The square-integrability of a representation extends to all its physical equivalence class. Thus, we can say consistently that a certain physical equivalence class of representations is square integrable.

⁷ Throughout the paper, we call a nonempty subset of a vector space V ‘linear span’ if it is a linear space itself (with respect to the operations of V), with no extra requirement of closedness with respect to any topology on V ; we prefer to use the term ‘(vector) subspace’ of V for indicating a *closed* linear span (with respect to a given topology on V).

- (ii) In the case where the l.c.s.c. group G is compact (hence, unimodular), every irreducible projective representation of G is square integrable (since, in this case, the Haar measure on G is finite) and, in the case of a unitary representation, theorem 1 coincides with the celebrated Peter–Weyl theorem.
- (iii) If the representation U of G is square integrable, then the orthogonality relations (100) imply that, for any nonzero admissible vector $\psi \in \mathcal{A}(U)$, one can define the linear operator

$$\mathfrak{W}_\psi : \mathcal{H} \ni \phi \mapsto \|\hat{D}_U \psi\|^{-1} c_{\psi, \phi}^U \in L^2(G) \tag{103}$$

—sometimes called (*generalized*) *wavelet transform* generated by U , with *analyzing* or *fiducial vector* ψ —which is an isometry. Note that \mathfrak{W}_ψ is the frame transform associated with the *normalized tight frame* $\{\|\hat{D}_U \psi\|^{-1} U(g)\psi\}_{g \in G}$ in \mathcal{H} based on (G, μ_G) . For the adjoint $\mathfrak{W}_\psi^* : L^2(G) \rightarrow \mathcal{H}$ of the isometry \mathfrak{W}_ψ the following formula holds (compare with the reconstruction formula (14)):

$$\mathfrak{W}_\psi^* f = \|\hat{D}_U \psi\|^{-1} \int_G f(g) (U(g)\psi) \, d\mu_G(g), \quad \forall f \in L^2(G). \tag{104}$$

The ordinary wavelet transform arises in the special case where G is the one-dimensional affine group $\mathbb{R} \rtimes \mathbb{R}_*^+$ (see [20]).

- (iv) The isometry \mathfrak{W}_ψ intertwines the square integrable representation U with the *left-regular m-representation* R_m of G in $L^2(G)$, see [22], which is the projective representation (with multiplier m) defined by

$$(R_m(g)f)(g') = \vec{m}(g, g') f(g^{-1}g'), \quad g, g' \in G, \tag{105}$$

$$\vec{m}(g, g') := m(g, g^{-1})^* m(g^{-1}, g'), \tag{106}$$

for every $f \in L^2(G)$; namely

$$\mathfrak{W}_\psi U(g) = R_m(g) \mathfrak{W}_\psi, \quad \forall g \in G. \tag{107}$$

Hence, U is (unitarily) equivalent to a subrepresentation of R_m . Note that, for $m \equiv 1$, $R \equiv R_m$ is the standard left regular representation of G .

- (v) Since \mathfrak{W}_ψ is a frame transform, the range $\mathcal{R}_\psi \equiv \text{Ran}(\mathfrak{W}_\psi)$ —which, by Schwarz inequality, consists of (equivalence classes of μ_G -almost everywhere) *bounded* square integrable functions—is a r.k.H.s. (embedded in $L^2(G)$; see remark 1), and the reproducing kernel is given explicitly by

$$\kappa_\psi^U(g, g') := \|\hat{D}_U \psi\|^{-2} \langle U(g)\psi, U(g')\psi \rangle, \quad g, g' \in G. \tag{108}$$

Namely, for every function f in \mathcal{R}_ψ , we have

$$f(g) = \int_G \kappa_\psi^U(g, g') f(g') \, d\mu_G(g'), \quad \text{for } \mu_G - a.a. \, g \in G. \tag{109}$$

- (vi) The wavelet transform \mathfrak{W}_ψ intertwines a bounded operator \hat{A} in \mathcal{H} with an integral operator in $L^2(G)$

$$\mathfrak{W}_\psi \hat{A} = \widehat{A}_\psi \mathfrak{W}_\psi, \quad \hat{A} \in \mathcal{B}(\mathcal{H}), \tag{110}$$

where

$$(\widehat{A}_\psi f)(g) := \int_G \kappa_\psi^U(\hat{A}; g, g') f(g') \, d\mu_G(g'), \quad f \in L^2(G), \tag{111}$$

$$\kappa_\psi^U(\hat{A}; g, g') := \|\hat{D}_U \psi\|^{-2} \langle U(g)\psi, \hat{A}U(g')\psi \rangle, \quad g, g' \in G; \tag{112}$$

in particular: $\kappa_\psi^U(I; g, g') = \kappa_\psi^U(g, g')$. Since

$$\kappa_\psi^U(\hat{A}; g, \cdot) = \|\hat{D}_U \psi\|^{-2} \langle U(\cdot) \psi, \hat{A}^* U(g) \psi \rangle^* \tag{113}$$

and the function $\|\hat{D}_U \psi\|^{-2} \langle U(\cdot) \psi, \hat{A}^* U(g) \psi \rangle$ belongs to \mathcal{R}_ψ , denoting by \mathcal{R}_ψ^\perp the orthogonal complement in $L^2(G)$ of \mathcal{R}_ψ , the operator \widehat{A}_ψ satisfies

$$\widehat{A}_\psi f = 0, \quad \forall f \in \mathcal{R}_\psi^\perp; \tag{114}$$

therefore, we have (compare with relation (19))

$$\widehat{A}_\psi = \mathfrak{W}_\psi \hat{A} \mathfrak{W}_\psi^*. \tag{115}$$

Moreover, relation (110) implies that $\hat{A} = \mathfrak{W}_\psi^* \widehat{A}_\psi \mathfrak{W}_\psi$; hence, by means of formulae (104) and (111), we get the following (weak integral) formula:

$$\hat{A} = \|\hat{D}_U \psi\|^{-2} \int_G d\mu_G(g) \int_G d\mu_G(g') \kappa_\psi^U(\hat{A}; g, g') |U(g) \psi\rangle \langle U(g') \psi|. \tag{116}$$

(vii) Since for the Fourier–Wigner transform a relation analogous to the Moyal identity holds true, namely,

$$\begin{aligned} \int_{\mathbb{R} \times \mathbb{R}} \mathcal{V}_{\phi_1 \psi_1} \widehat{\mathcal{V}}_{\phi_2 \psi_2}^*(q, p) \mathcal{V}_{\phi_2 \psi_2}(q, p) dq dp &= \int_{\mathbb{R} \times \mathbb{R}} \langle \phi_1, U(q, p) \psi_1 \rangle \langle U(q, p) \psi_2, \phi_2 \rangle \frac{dq dp}{(2\pi)^2} \\ &= \frac{1}{2\pi} \langle \phi_1, \phi_2 \rangle \langle \psi_2, \psi_1 \rangle, \end{aligned} \tag{117}$$

we conclude that the projective representation $U : \mathbb{R} \times \mathbb{R} \ni (q, p) \mapsto \exp(i(p\hat{q} - q\hat{p})) \in \mathcal{U}(L^2(\mathbb{R}))$ is square integrable and, fixing $(2\pi)^{-1} dq dp$ as the Haar measure on $\mathbb{R} \times \mathbb{R}$, we have that $\hat{D}_U = I$. Therefore, the Haar measure $(2\pi)^{-1} dq dp$ is normalized in agreement with U . If $\psi \in L^2(\mathbb{R})$ is the ground state of the quantum harmonic oscillator, then $\{U(q, p) \psi\}_{q, p \in \mathbb{R}}$ is the family of standard *coherent states* [23, 48], which is a normalized tight frame in $L^2(\mathbb{R})$ based on $(\mathbb{R} \times \mathbb{R}, (2\pi)^{-1} dq dp)$.

As a consequence of the ‘trace formula for frames’—see proposition 1—we have the following further remarkable property of square integrable representations:

Proposition 6 (the ‘first trace form. for sq. int. reps.’). *Let $U : G \rightarrow \mathcal{U}(\mathcal{H})$ be a square integrable projective representation and \hat{D}_U the associated Duflo–Moore operator (normalized according to the left Haar measure μ_G). Then, for any couple of admissible vectors $\psi, \phi \in \mathcal{A}(U)$ and any trace class operator \hat{A} in \mathcal{H} , the following formula holds:*

$$\text{tr}(\hat{A}) \langle \hat{D}_U \psi, \hat{D}_U \phi \rangle = \int_G \langle U(g) \psi, \hat{A} U(g) \phi \rangle d\mu_G(g). \tag{118}$$

Proof. We will assume that $\psi \neq 0 \neq \phi$, otherwise the statement is trivial; we will further assume, for the moment, that $\phi = \psi \in \mathcal{A}(U)$. Then, as already observed, the set of vectors $\{\|\hat{D}_U \psi\|^{-1} U(g) \psi\}_{g \in G}$ is a normalized tight frame in \mathcal{H} based on (G, μ_G) , and formula (118)—for every $\hat{A} \in \mathcal{B}_1(\mathcal{H})$ and with $\phi = \psi$ —follows from formula (21) applied to this frame.

In order to extend the proof to the case where $\phi \neq \psi$, we can use the result just obtained and a standard ‘polarization argument’. The proof is complete. \square

One can furthermore prove that, in the case where the l.c.s.c. group G is unimodular, the first trace formula for square integrable representations is a particular case of the following result:

Proposition 7 (the ‘second trace form. for sq. int. reps.’). *Let $U : G \rightarrow \mathcal{U}(\mathcal{H})$ be a square integrable projective representation of a unimodular l.c.s.c. group G and let $\hat{D}_U = d_U I$, $d_U > 0$, be the associated Duflo–Moore operator (normalized according to the Haar measure μ_G). Then, for any couple of trace class operators \hat{A}, \hat{T} in \mathcal{H} , the following formula holds:*

$$\text{tr}(\hat{A}) \text{tr}(\hat{T}) = d_U^{-2} \int_G \text{tr}(U(g)\hat{T}U(g)^*\hat{A}) \, d\mu_G(g). \tag{119}$$

Proof. As in the proof of proposition 1, we can exploit the fact that every trace class operator can be expressed as a linear combination of four *positive* trace class operators, and we can restrict the proof of relation (119)—with no loss of generality—to the case where \hat{A}, \hat{T} are generic nonzero *positive* trace class operator in \mathcal{H} . Then, let us consider the canonical decomposition of \hat{T} as a nonzero (positive) compact operator

$$\hat{T} = \sum_{n \in \mathcal{N}} \tau_n |\psi_n\rangle\langle\psi_n|, \quad \psi_n, \phi_n \in \mathcal{H}, \tag{120}$$

where \mathcal{N} is a finite or countably infinite index set, $\{\psi_n\}_{n \in \mathcal{N}}$ is an orthonormal system and $\{\tau_n\}_{n \in \mathcal{N}}$ is a set of strictly positive numbers—the nonzero singular values of \hat{T} (which, being \hat{T} positive, coincide with the nonzero eigenvalues of \hat{T})—such that $\sum_{n \in \mathcal{N}} \tau_n = \text{tr}(\hat{T})$; the sum (120) converges with respect to the trace norm. Observe that the map

$$\mathcal{B}_1(\mathcal{H}) \ni \hat{S} \mapsto \text{tr}(U(g)\hat{S}U(g)^*\hat{A}) = \text{tr}(\hat{S}U(g)^*\hat{A}U(g)) \in \mathbb{C} \tag{121}$$

is a bounded linear functional; hence

$$\text{tr}(U(g)\hat{T}U(g)^*\hat{A}) = \sum_{n \in \mathcal{N}} \tau_n \text{tr}(U(g)|\psi_n\rangle\langle\psi_n|U(g)^*\hat{A}). \tag{122}$$

Therefore, we have

$$\begin{aligned} \int_G \text{tr}(U(g)\hat{T}U(g)^*\hat{A}) \, d\mu_G(g) &= \int_G \sum_{n \in \mathcal{N}} \tau_n \langle U(g)\psi_n, \hat{A}U(g)\psi_n \rangle \, d\mu_G(g) \\ &= \sum_{n \in \mathcal{N}} \tau_n \int_G \langle U(g)\psi_n, \hat{A}U(g)\psi_n \rangle \, d\mu_G(g) \\ &= d_U^2 \text{tr}(\hat{A}) \text{tr}(\hat{T}), \end{aligned} \tag{123}$$

where the permutation of the (possibly infinite) sum with the integral is allowed by the positivity of the integrand functions and we have used the first trace formula (118). \square

In the following section, it will be shown that the notion of square integrable representation allows us to give a rigorous definition of the (standard) Wigner transform, and to generalize this definition in a straightforward way: with every square integrable projective representation one can associate a suitable isometry, i.e. a (generalized) Wigner transform.

5. Wigner transforms associated with square integrable representations and the Wigner distribution

The (generalized) wavelet transform defined in the previous section is not the only remarkable linear map that one can construct, in a natural way, by means of a square integrable representation. Indeed, following [12], we will show that—given a square integrable projective representation $U : G \rightarrow \mathcal{U}(\mathcal{H})$ (with multiplier m)—with every Hilbert–Schmidt operator $\hat{A} \in \mathcal{B}_2(\mathcal{H})$ one can suitably associate a function $G \ni g \mapsto (\mathfrak{S}_U \hat{A})(g) \in \mathbb{C}$ contained

in $L^2(G) \equiv L^2(G, \mu_G; \mathbb{C})$. Denoting by \hat{D}_U , as in section 4, the Duflo–Moore operator associated with U (normalized according to a left Haar measure μ_G on G), *formally* we set

$$(\mathfrak{S}_U \hat{A})(g) := \text{tr}(U(g)^* \hat{A} \hat{D}_U^{-1}). \quad (124)$$

Since the operator $U(g)^* \hat{A} \hat{D}_U^{-1}$ (or, possibly, its closure) is not, in general, a trace class operator, definition (124) is meaningless unless we provide a rigorous interpretation. To this aim, we will exploit the fact that finite-rank operators form a dense linear span $\text{FR}(\mathcal{H})$ in $\mathcal{B}_2(\mathcal{H})$. Precisely, consider those rank one operators in \mathcal{H} that are of the type

$$\widehat{\phi\psi} = |\phi\rangle\langle\psi|, \quad \phi \in \mathcal{H}, \quad \psi \in \text{Dom}(\hat{D}_U^{-1}). \quad (125)$$

The linear span generated by the operators of this form, namely the set

$$\text{FR}(\mathcal{H}; U) := \{\hat{F} \in \text{FR}(\mathcal{H}) : \text{Ran}(\hat{F}^*) \subset \text{Dom}(\hat{D}_U^{-1})\}, \quad (126)$$

is dense in $\text{FR}(\mathcal{H})$; hence, in $\mathcal{B}_2(\mathcal{H})$: $\overline{\text{FR}(\mathcal{H}; U)} = \mathcal{B}_2(\mathcal{H})$. Observe, moreover, that if we set

$$(\mathfrak{S}_U \widehat{\phi\psi})(g) := \text{tr}(U(g)^* |\phi\rangle\langle\hat{D}_U^{-1}\psi|) = \langle U(g) \hat{D}_U^{-1}\psi, \phi \rangle, \quad \forall \widehat{\phi\psi} \in \text{FR}(\mathcal{H}; U), \quad (127)$$

then, by virtue of the orthogonality relations (100), for any $\widehat{\phi_1\psi_1}, \widehat{\phi_2\psi_2} \in \text{FR}(\mathcal{H}; U)$ we have

$$\begin{aligned} \int_G (\mathfrak{S}_U \widehat{\phi_1\psi_1})(g)^* (\mathfrak{S}_U \widehat{\phi_2\psi_2})(g) \, d\mu_G(g) &= \int_G \langle \phi_1, U(g) \hat{D}_U^{-1}\psi_1 \rangle \langle U(g) \hat{D}_U^{-1}\psi_2, \phi_2 \rangle \, d\mu_G(g) \\ &= \langle \phi_1, \phi_2 \rangle \langle \psi_2, \psi_1 \rangle = \langle \widehat{\phi_1\psi_1}, \widehat{\phi_2\psi_2} \rangle_{\mathcal{B}_2(\mathcal{H})}. \end{aligned} \quad (128)$$

Therefore, extending the map \mathfrak{S}_U to all $\text{FR}(\mathcal{H}; U)$ by linearity, and then to the whole Hilbert space $\mathcal{B}_2(\mathcal{H})$ by continuity, we obtain an *isometry* $\mathfrak{S}_U : \mathcal{B}_2(\mathcal{H}) \rightarrow L^2(G)$ called the (*generalized*) *Wigner transform* generated by U . As the reader may check, if the group G is unimodular ($\Rightarrow \hat{D}_U = d_U I$, with $d_U > 0$), then for every trace class operator $\hat{\rho} \in \mathcal{B}_1(\mathcal{H})$ —in particular, for every density operator in \mathcal{H} —we have simply

$$(\mathfrak{S}_U \hat{\rho})(g) = d_U^{-1} \text{tr}(U(g)^* \hat{\rho}). \quad (129)$$

Let us now investigate the intertwining property of the isometry \mathfrak{S}_U with respect to the natural action of the group G in $\mathcal{B}_2(\mathcal{H})$. Precisely, let us consider the map $U \vee U : G \rightarrow \mathcal{U}(\mathcal{B}_2(\mathcal{H}))$ defined by

$$U \vee U(g) \hat{A} := U(g) \hat{A} U(g)^*, \quad \forall g \in G, \quad \hat{A} \in \mathcal{B}_2(\mathcal{H}). \quad (130)$$

The map $U \vee U$ is a (strongly continuous) *unitary* representation—even if, in general, the representation U has only been assumed to be *projective*—which can be regarded as the standard action of the ‘symmetry group’ G on the quantum ‘observables’ (or on the ‘states’). Next, let us consider the map $\mathcal{T}_m : G \rightarrow \mathcal{U}(L^2(G))$ defined by

$$(\mathcal{T}_m(g)f)(g') = \Delta_G(g)^{\frac{1}{2}} \overleftrightarrow{m}(g, g') f(g^{-1}g'), \quad (131)$$

where the function $\overleftrightarrow{m} : G \times G \rightarrow \mathbb{T}$ has the following expression:

$$\overleftrightarrow{m}(g, g') := m(g, g^{-1}g')^* m(g^{-1}g', g). \quad (132)$$

As the reader may check by means of a direct calculation involving multipliers, the map \mathcal{T}_m is a unitary representation; the presence of the square root of the modular function Δ_G in formula (131) takes into account the right action of G on itself. Note that, for $m \equiv 1$, it coincides with the restriction to the ‘diagonal subgroup’ of the *two-sided regular representation* of the direct product group $G \times G$; see [47, 49]. As the reader may check using relation (102), the Wigner transform \mathfrak{S}_U intertwines the representations $U \vee U$ and \mathcal{T}_m

$$\mathfrak{S}_U U \vee U(g) = \mathcal{T}_m(g) \mathfrak{S}_U, \quad \forall g \in G. \quad (133)$$

Since the generalized Wigner transform \mathfrak{S}_U is an isometry, the adjoint map $\mathfrak{S}_U^* : L^2(G) \rightarrow \mathcal{B}_2(\mathcal{H})$ is a partial isometry such that

$$\mathfrak{S}_U^* \mathfrak{S}_U = I, \quad \mathfrak{S}_U \mathfrak{S}_U^* = \hat{P}_{\mathcal{R}_U}, \tag{134}$$

where $\hat{P}_{\mathcal{R}_U}$ is the orthogonal projection onto the subspace $\mathcal{R}_U \equiv \text{Ran}(\mathfrak{S}_U) = \text{Ker}(\mathfrak{S}_U^*)$ of $L^2(G)$. Thus, the partial isometry \mathfrak{S}_U^* is the pseudo-inverse of \mathfrak{S}_U and we will call it (*generalized*) *Weyl map* associated with the representation U . It is remarkable that the Weyl map \mathfrak{S}_U^* admits the following weak integral expression (see [12]):

$$\mathfrak{S}_U^* f = \int_G f(g) U(g) \hat{D}_U^{-1} d\mu_G(g), \quad \forall f \in L^2(G). \tag{135}$$

Observe that, in the case where the group G is unimodular, with the Haar measure μ_G normalized in agreement with U , we have simply

$$\mathfrak{S}_U^* f = \int_G f(g) U(g) d\mu_G(g), \quad \forall f \in L^2(G). \tag{136}$$

Let us now focus on the case where $G = \mathbb{R} \times \mathbb{R}$ and U is the square integrable projective representation

$$U : \mathbb{R} \times \mathbb{R} \ni (q, p) \mapsto \exp(i(p\hat{q} - q\hat{p})) \in \mathcal{U}(L^2(\mathbb{R})). \tag{137}$$

We recall from section 4 that $(2\pi)^{-1} dq dp$ is the Haar measure on $\mathbb{R} \times \mathbb{R}$ normalized in agreement with U . Then, in this case, the generalized Wigner transform \mathfrak{S}_U is the isometry from $\mathcal{B}_2(L^2(\mathbb{R}))$ into $L^2(\mathbb{R} \times \mathbb{R}) \equiv L^2(\mathbb{R} \times \mathbb{R}, (2\pi)^{-1} dq dp; \mathbb{C})$ determined by

$$(\mathfrak{S}_U \hat{\rho})(q, p) = \text{tr}(U(q, p)^* \hat{\rho}), \quad \forall \hat{\rho} \in \mathcal{B}_1(L^2(\mathbb{R})). \tag{138}$$

For a pure state $\hat{\psi} \equiv |\psi\rangle\langle\psi| \in \mathcal{B}_2(L^2(\mathbb{R}))$, $\|\psi\| = 1$, the function $\mathfrak{S}_U \hat{\psi}$ coincides—up to an irrelevant normalization factor—with the Fourier–Wigner distribution associated with $\hat{\psi}$ (compare with definition (93)). The multiplier $m : (\mathbb{R} \times \mathbb{R}) \times (\mathbb{R} \times \mathbb{R}) \rightarrow \mathbb{T}$ associated with U is given by

$$m(q, p; q', p') = \exp\left(\frac{i}{2}(qp' - pq')\right). \tag{139}$$

Hence, for the function \overleftrightarrow{m} we find, in this case, the following expression:

$$\overleftrightarrow{m}(q, p; q', p') = m(q, p; q' - q, p' - p)^* m(q' - q, p' - p; q, p) = \exp(-i(qp' - pq')). \tag{140}$$

Recalling formula (131), we conclude that the generalized Wigner transform \mathfrak{S}_U intertwines the unitary representation $U \vee U : \mathbb{R} \times \mathbb{R} \rightarrow \mathcal{U}(\mathcal{B}_2(L^2(\mathbb{R})))$ with the representation $\mathcal{T}_m : \mathbb{R} \times \mathbb{R} \rightarrow \mathcal{U}(L^2(\mathbb{R} \times \mathbb{R}))$ defined by

$$(\mathcal{T}_m(q, p) f)(q', p') = e^{-i(qp' - pq')} f(q', p'), \quad \forall f \in L^2(\mathbb{R} \times \mathbb{R}). \tag{141}$$

The *standard Wigner transform*—we will denote it by \mathfrak{T} —is the isometry obtained composing the isometry \mathfrak{S}_U determined by (138) with the symplectic Fourier transform

$$\mathfrak{F} := \mathcal{F}_{sp} \mathfrak{S}_U : \mathcal{B}_2(L^2(\mathbb{R})) \rightarrow L^2(\mathbb{R} \times \mathbb{R}). \tag{142}$$

In particular, for a pure state $\hat{\psi} \in \mathcal{B}_2(L^2(\mathbb{R}))$ the function $\mathfrak{T} \hat{\psi}$ coincides, up to an irrelevant normalization factor, with the Wigner distribution associated with $\hat{\psi}$ (compare with formula (90))

$$(\mathfrak{T} \hat{\psi})(q, p) = 2\pi Q_{\hat{\psi}}(q, p). \tag{143}$$

It is clear that the isometry \mathfrak{T} intertwines the representation $U \vee U$ with the unitary representation $\mathcal{T} : \mathbb{R} \times \mathbb{R} \rightarrow \mathcal{U}(L^2(\mathbb{R} \times \mathbb{R}))$ defined by

$$\mathcal{T}(q, p) = \mathcal{F}_{sp} \mathcal{T}_m(q, p) \mathcal{F}_{sp}, \quad \forall (q, p) \in \mathbb{R} \times \mathbb{R}; \tag{144}$$

as the reader may easily check, explicitly, we have

$$(\mathcal{T}(q, p)f)(q', p') = f(q' - q, p' - p), \quad \forall f \in L^2(\mathbb{R} \times \mathbb{R}). \tag{145}$$

Note that this result is consistent with relations (83) and (84). It is also a remarkable result—see [50]—that $\text{Ran}(\mathfrak{S}_U) = \text{Ran}(\mathfrak{T}) = L^2(\mathbb{R} \times \mathbb{R})$. Therefore, the standard Wigner transform \mathfrak{T} —and its adjoint \mathfrak{T}^* , the *standard Weyl map*—are both unitary operators.

Note that, according to the definition of the map \mathfrak{S}_U , the Wigner transform associated with a square integrable representation is not—in general—a frame transform. For instance, in the case where U is the Weyl system (137), it is not. This is coherent with the fact that, in the mentioned case, $\text{Ran}(\mathfrak{S}_U) = L^2(\mathbb{R} \times \mathbb{R})$ and hence $\text{Ran}(\mathfrak{S}_U)$ is not a r.k.H.s. as it should be if \mathfrak{S}_U were a frame transform. For the same reason, the standard Wigner transform \mathfrak{T} is not a frame transform. It is then natural to address the following problem: given a square integrable projective representation U , is it possible to associate with U , in a straightforward way, a frame transform in $\mathcal{B}_2(\mathcal{H})$? We will give an (affirmative) answer to this question in the subsequent section.

6. Frames in Hilbert–Schmidt spaces from square integrable representations

In this section, we will show that it is possible to obtain from a square integrable representation—in a natural way—frame transforms having as domain the space of Hilbert–Schmidt operators in the Hilbert space where the representation acts. In the following, we will assume that G is a l.c.s.c. group and $U : G \rightarrow \mathcal{U}(\mathcal{H})$ a square integrable projective representation of G in the Hilbert space \mathcal{H} . For the sake of simplicity, we will suppose that the group G is *unimodular*, but the results that we are going to prove actually extend to the general case (see remark 10 below). We will denote by μ_G the Haar measure on G *normalized in agreement with the representation U* (see remark 4). Now, for any couple of Hilbert–Schmidt operators $\hat{A}, \hat{T} \in \mathcal{B}_2(\mathcal{H})$, we can define the function

$$A : G \times G \ni (g_1, g_2) \mapsto \langle \hat{T}(g_1, g_2), \hat{A} \rangle_{\mathcal{B}_2(\mathcal{H})} \in \mathbb{C}, \tag{146}$$

where

$$\hat{T}(g_1, g_2) := U(g_1) \hat{T} U(g_2)^*, \quad g_1, g_2 \in G. \tag{147}$$

At this point, we have the following result:

Theorem 2. *With the previous notations and assumptions, for any $\hat{A}, \hat{T} \in \mathcal{B}_2(\mathcal{H})$, the map*

$$\langle \hat{T}(\cdot, \cdot), \hat{A} \rangle_{\mathcal{B}_2(\mathcal{H})} : G \times G \ni (g_1, g_2) \mapsto \langle \hat{T}(g_1, g_2), \hat{A} \rangle_{\mathcal{B}_2(\mathcal{H})} \in \mathbb{C} \tag{148}$$

is a Borel function contained in $L^2(G \times G) \equiv L^2(G \times G, \mu_G \otimes \mu_G; \mathbb{C})$, and the linear application

$$\mathfrak{D}_{\hat{T}} : \mathcal{B}_2(\mathcal{H}) \ni \hat{A} \mapsto A = \langle \hat{T}(\cdot, \cdot), \hat{A} \rangle_{\mathcal{B}_2(\mathcal{H})} \in L^2(G \times G) \tag{149}$$

—for \hat{T} nonzero and normalized (i.e. $\|\hat{T}\|_{\mathcal{B}_2(\mathcal{H})} = 1$)—is an isometry (the ‘dequantization map’ associated with the representation U , with ‘analyzing operator’ \hat{T}); namely, for \hat{T} normalized, the family of operators $\{\hat{T}(g_1, g_2) : (g_1, g_2) \in G \times G\}$ is a normalized tight frame in $\mathcal{B}_2(\mathcal{H})$, based on $(G \times G, \mu_G \otimes \mu_G)$. Moreover, for any $\hat{A}, \hat{B}, \hat{S}, \hat{T} \in \mathcal{B}_2(\mathcal{H})$, the following relation holds:

$$\int_{G \times G} (\mathfrak{D}_{\hat{T}} \hat{A})(g_1, g_2)^* (\mathfrak{D}_{\hat{S}} \hat{B})(g_1, g_2) d\mu_G \otimes \mu_G(g_1, g_2) = \langle \hat{A}, \hat{B} \rangle_{\mathcal{B}_2(\mathcal{H})} \langle \hat{S}, \hat{T} \rangle_{\mathcal{B}_2(\mathcal{H})}. \tag{150}$$

Proof. Let \hat{T} be a nonzero operator in $\mathcal{B}_2(\mathcal{H})$. As a Hilbert–Schmidt operator, \hat{T} will admit a canonical decomposition of the form

$$\hat{T} = \sum_{n \in \mathcal{N}} \tau_n |\phi_n\rangle \langle \psi_n|, \quad \psi_n, \phi_n \in \mathcal{H}, \tag{151}$$

where \mathcal{N} is a finite or countably infinite index set, $\{\psi_n\}_{n \in \mathcal{N}}, \{\phi_n\}_{n \in \mathcal{N}}$ are orthonormal systems and $\{\tau_n\}_{n \in \mathcal{N}}$ is a set of strictly positive numbers (the nonzero singular values of \hat{T}) such that $\sum_{n \in \mathcal{N}} \tau_n^2 = \|\hat{T}\|_{\mathcal{B}_2(\mathcal{H})}^2$; the sum (151) converges with respect to the Hilbert–Schmidt norm.

The fact that the representation U is a weakly Borel map implies that the function $\langle \hat{T}(\cdot, \cdot), \hat{A} \rangle_{\mathcal{B}_2(\mathcal{H})}$ —for any $\hat{A}, \hat{T} \in \mathcal{B}_2(\mathcal{H})$ —is Borel; namely, that the application $G \times G \ni (g_1, g_2) \mapsto \hat{T}(g_1, g_2) \in \mathcal{B}_2(\mathcal{H})$ is weakly Borel. In fact, by means of the canonical decompositions of the operators \hat{A} and \hat{T} , one can express the function $\langle \hat{T}(\cdot, \cdot), \hat{A} \rangle_{\mathcal{B}_2(\mathcal{H})}$ as a finite—or countably infinite and pointwise converging—sum of Borel functions; we leave the details to the reader (recall that, given Borel functions $f_j : G \rightarrow \mathbb{C}, j = 1, 2$, the function $f : G \times G \ni (g_1, g_2) \mapsto f_1(g_1)f_2(g_2) \in \mathbb{C}$ is Borel too).

Assume, now, that $\hat{T} \neq 0$ and $\|\hat{T}\|_{\mathcal{B}_2(\mathcal{H})} = 1$, and let \hat{A} be an arbitrary operator in $\mathcal{B}_2(\mathcal{H})$. Consider the associated Borel complex-valued function $A \equiv \langle \hat{T}(\cdot, \cdot), \hat{A} \rangle_{\mathcal{B}_2(\mathcal{H})}$ on $G \times G$. We will prove that this function belongs to $L^2(G \times G)$ and, simultaneously, that the dequantization map (149) is an isometry. To this aim, it will be convenient to assume for the moment that \hat{T} is a *finite-rank operator*; this is equivalent to suppose that the index set \mathcal{N} is *finite*. Then, by Tonelli’s theorem and the (finite) canonical decomposition of \hat{T} , we have

$$\begin{aligned} \int_{G \times G} |A(g_1, g_2)|^2 d\mu_G \otimes \mu_G(g_1, g_2) &= \int_G \left(\int_G |A(g_1, g_2)|^2 d\mu_G(g_2) \right) d\mu_G(g_1) \\ &= \sum_{n, k \in \mathcal{N}} \tau_n \tau_k \int_G \left(\int_G \langle \hat{A}, (|\phi_n(g_1)\rangle \langle \psi_n(g_2)|) \rangle_{\mathcal{B}_2(\mathcal{H})} \right. \\ &\quad \left. \times \langle (|\phi_k(g_1)\rangle \langle \psi_k(g_2)|), \hat{A} \rangle_{\mathcal{B}_2(\mathcal{H})} d\mu_G(g_1) \right) d\mu_G(g_2), \end{aligned} \tag{152}$$

where, for the sake of notational conciseness, we have set

$$\phi_n(g) := U(g)\phi_n, \quad \psi_n(g) = U(g)\psi_n, \quad \forall g \in G, \quad \forall n \in \mathcal{N}. \tag{153}$$

Next, observe that

$$\langle \hat{A}, (|\phi_n(g_1)\rangle \langle \psi_n(g_2)|) \rangle_{\mathcal{B}_2(\mathcal{H})} = \text{tr}(|\psi_n(g_2)\rangle \langle \phi_n(g_1)| \hat{A})^* = \langle \hat{A}\psi_n(g_2), \phi_n(g_1) \rangle; \tag{154}$$

hence, from relations (152) and (154), we obtain

$$\begin{aligned} \int_{G \times G} |A(g_1, g_2)|^2 d\mu_G \otimes \mu_G(g_1, g_2) &= \sum_{n, k \in \mathcal{N}} \tau_n \tau_k \int_G \left(\int_G \langle \hat{A}\psi_n(g_2), \phi_n(g_1) \rangle \right. \\ &\quad \left. \times \langle \phi_k(g_1), \hat{A}\psi_k(g_2) \rangle d\mu_G(g_1) \right) d\mu_G(g_2) \\ &= \sum_{n \in \mathcal{N}} \tau_n^2 \int_G \langle \hat{A}U(g_2)\psi_n, \hat{A}U(g_2)\psi_n \rangle d\mu_G(g_2), \end{aligned} \tag{155}$$

where we have used the orthogonality relations for the square integrable representation U (G unimodular, μ_G normalized in agreement with U). At this point, using the trace formula (118), we get

$$\int_{G \times G} |A(g_1, g_2)|^2 d\mu_G \otimes \mu_G(g_1, g_2) = \text{tr}(\hat{A}^* \hat{A}) \sum_{n \in \mathcal{N}} \tau_n^2 = \|\hat{A}\|_{\mathcal{B}_2(\mathcal{H})}^2 \|\hat{T}\|_{\mathcal{B}_2(\mathcal{H})}^2, \tag{156}$$

with $\|\hat{T}\|_{\mathcal{B}_2(\mathcal{H})}^2 = 1$. Thus, in the case where the index set \mathcal{N} is finite, the proof is complete.

Suppose now that $\dim(\mathcal{H}) = \infty$ and $\mathcal{N} = \mathbb{N}$. In this case, we can consider a sequence $\{\hat{T}_N\}_{N \in \mathbb{N}} \subset \mathcal{B}_2(\mathcal{H})$ of *finite-rank operators* converging to \hat{T} : $\lim_{N \rightarrow \infty} \|\hat{T} - \hat{T}_N\|_{\mathcal{B}_2(\mathcal{H})} = 0$; in particular, we can consider the sequence of finite truncations of the canonical decomposition of \hat{T} , i.e. $\hat{T}_N := \sum_{n=1}^N \tau_n |\psi_n\rangle\langle\phi_n|$. Then, setting $\hat{T}_N(g_1, g_2) := U(g_1)\hat{T}_N U(g_2)^*$, we get

$$\lim_{N \rightarrow \infty} \|\hat{T}(g_1, g_2) - \hat{T}_N(g_1, g_2)\|_{\mathcal{B}_2(\mathcal{H})} = \lim_{N \rightarrow \infty} \|\hat{T} - \hat{T}_N\|_{\mathcal{B}_2(\mathcal{H})} = 0, \quad (157)$$

and

$$A(g_1, g_2) := \langle \hat{T}(g_1, g_2), \hat{A} \rangle_{\mathcal{B}_2(\mathcal{H})} = \lim_{N \rightarrow \infty} \langle \hat{T}_N(g_1, g_2), \hat{A} \rangle_{\mathcal{B}_2(\mathcal{H})}, \quad \forall g_1, g_2 \in G. \quad (158)$$

Next, observe that for every $N \in \mathbb{N}$ the function $A_N := \langle \hat{T}_N(\cdot, \cdot), \hat{A} \rangle_{\mathcal{B}_2(\mathcal{H})} : G \times G \rightarrow \mathbb{C}$ belongs to $L^2(G \times G)$, and $\{A_N\}_{N \in \mathbb{N}}$ is a Cauchy sequence in $L^2(G \times G)$. Indeed—according to the first segment of the proof—one finds out that, for any $N, N' \in \mathbb{N}$,

$$\begin{aligned} \int_{G \times G} |A_{N'}(g_1, g_2) - A_N(g_1, g_2)|^2 d\mu_G \otimes \mu_G(g_1, g_2) &= \|\langle \hat{T}_{N,N'}(\cdot, \cdot), \hat{A} \rangle_{\mathcal{B}_2(\mathcal{H})}\|_{L^2(G \times G)}^2 \\ &= \|\hat{A}\|_{\mathcal{B}_2(\mathcal{H})}^2 \|\hat{T}_{N'} - \hat{T}_N\|_{\mathcal{B}_2(\mathcal{H})}^2, \end{aligned} \quad (159)$$

where we have set $\hat{T}_{N,N'}(g_1, g_2) \equiv U(g_1)(\hat{T}_{N'} - \hat{T}_N)U(g_2)^*$, and we have exploited the fact that $\hat{T}_{N'} - \hat{T}_N$ is a finite-rank operator. Therefore, the function $A : G \times G \rightarrow \mathbb{C}$ is the pointwise limit of a Cauchy sequence of functions in $L^2(G \times G)$, so that—according to a well-known result—it belongs to $L^2(G \times G)$ too and $\lim_{N \rightarrow \infty} \|A - A_N\|_{L^2(G \times G)} = 0$. Hence, considering that $\|A_N\|_{L^2(G \times G)} = \|\hat{A}\|_{\mathcal{B}_2(\mathcal{H})} \|\hat{T}_N\|_{\mathcal{B}_2(\mathcal{H})}$, we have

$$\begin{aligned} \|A\|_{L^2(G \times G)} &= \lim_{N \rightarrow \infty} \|A_N\|_{L^2(G \times G)} \\ &= \|\hat{A}\|_{\mathcal{B}_2(\mathcal{H})} \lim_{N \rightarrow \infty} \|\hat{T}_N\|_{\mathcal{B}_2(\mathcal{H})} = \|\hat{A}\|_{\mathcal{B}_2(\mathcal{H})} \|\hat{T}\|_{\mathcal{B}_2(\mathcal{H})}, \end{aligned} \quad (160)$$

with $\|\hat{T}\|_{\mathcal{B}_2(\mathcal{H})} = 1$. Thus, the first part of the proof is complete.

We will now prove relation (150). This second part of the proof goes along lines similar to the ones already traced in the first part, so we will be rather sketchy.

Let $\hat{A}, \hat{B}, \hat{S}, \hat{T}$ be operators in $\mathcal{B}_2(\mathcal{H})$, with $\hat{S} \neq 0 \neq \hat{T}$ (otherwise relation (150) is trivial), and consider the canonical decompositions

$$\hat{S} = \sum_{m \in \mathcal{M}} \sigma_m |\eta_m\rangle\langle\chi_m|, \quad \hat{T} = \sum_{n \in \mathcal{N}} \tau_n |\phi_n\rangle\langle\psi_n|, \quad \eta_m, \chi_m, \psi_n, \phi_n \in \mathcal{H}, \quad (161)$$

where \mathcal{M}, \mathcal{N} are finite or countably infinite index sets, $\{\eta_m\}_{m \in \mathcal{M}}, \{\chi_m\}_{m \in \mathcal{M}}, \{\psi_n\}_{n \in \mathcal{N}}, \{\phi_n\}_{n \in \mathcal{N}}$ orthonormal systems, $\{\sigma_m\}_{m \in \mathcal{M}}$ and $\{\tau_n\}_{n \in \mathcal{N}}$ sets of strictly positive numbers such that $\sum_{m \in \mathcal{M}} \sigma_m^2 = \|\hat{S}\|_{\mathcal{B}_2(\mathcal{H})}^2$ and $\sum_{n \in \mathcal{N}} \tau_n^2 = \|\hat{T}\|_{\mathcal{B}_2(\mathcal{H})}^2$, and we have

$$\langle \hat{S}, \hat{T} \rangle_{\mathcal{B}_2(\mathcal{H})} = \sum_{m \in \mathcal{M}} \sum_{n \in \mathcal{N}} \sigma_m \tau_n \langle \eta_m, \phi_n \rangle \langle \psi_n, \chi_m \rangle. \quad (162)$$

The sums (161) converge with respect to the Hilbert–Schmidt norm.

Suppose first that the index sets \mathcal{M}, \mathcal{N} are both *finite*. For notational conciseness, we define the function $\Phi : G \times G \rightarrow \mathbb{C}$, $\Phi(g_1, g_2) := (\mathfrak{D}_{\hat{T}} \hat{A})(g_1, g_2)^* (\mathfrak{D}_{\hat{S}} \hat{B})(g_1, g_2)$, and we set

$$\eta_m(g) := U(g)\eta_m, \quad \chi_m(g) := U(g)\chi_m, \quad \phi_n(g) := U(g)\phi_n, \quad \psi_n(g) := U(g)\psi_n. \quad (163)$$

Then, since the function Φ belongs to $L^1(G \times G)$ (according to the first part of the proof), we can apply Fubini's theorem thus getting

$$\begin{aligned} \int_{G \times G} \Phi(g_1, g_2) d\mu_G \otimes \mu_G(g_1, g_2) &= \sum_{m \in \mathcal{M}} \sum_{n \in \mathcal{N}} \sigma_m \tau_n \int_G \left(\int_G \langle \hat{A} \psi_n(g_2), \phi_n(g_1) \rangle \right. \\ &\quad \left. \times \langle \eta_m(g_1), \hat{B} \chi_m(g_2) \rangle d\mu_G(g_1) \right) d\mu_G(g_2) \\ &= \sum_{m \in \mathcal{M}} \sum_{n \in \mathcal{N}} \sigma_m \tau_n \langle \eta_m, \phi_n \rangle \\ &\quad \times \int_G \langle \hat{A} U(g_2) \psi_n, \hat{B} U(g_2) \chi_m \rangle d\mu_G(g_2), \end{aligned} \tag{164}$$

where we have used the orthogonality relations for U . Next, use the trace formula (118)

$$\begin{aligned} \int_{G \times G} \Phi(g_1, g_2) d\mu_G \otimes \mu_G(g_1, g_2) &= \sum_{m \in \mathcal{M}} \sum_{n \in \mathcal{N}} \sigma_m \tau_n \langle \eta_m, \phi_n \rangle \\ &\quad \times \int_G \langle U(g_2) \psi_n, \hat{A}^* \hat{B} U(g_2) \chi_m \rangle d\mu_G(g_2) \\ &= \sum_{m \in \mathcal{M}} \sum_{n \in \mathcal{N}} \sigma_m \tau_n \langle \eta_m, \phi_n \rangle \langle \psi_n, \chi_m \rangle \text{tr}(\hat{A}^* \hat{B}) \\ &= \langle \hat{A}, \hat{B} \rangle_{\mathcal{B}_2(\mathcal{H})} \langle \hat{S}, \hat{T} \rangle_{\mathcal{B}_2(\mathcal{H})}. \end{aligned} \tag{165}$$

Suppose now that $\dim(\mathcal{H}) = \infty$, and that $\mathcal{M} = \mathbb{N}$ and/or $\mathcal{N} = \mathbb{N}$. Then, one can adopt a reasoning similar to the one used in the second half of the first part of the proof: consider sequences $\{\hat{S}_M\}_{M \in \mathbb{N}}$ and/or $\{\hat{T}_N\}_{N \in \mathbb{N}}$ of finite-rank operators—converging to \hat{S} and/or to \hat{T} , respectively—and exploit the continuity (in both arguments) of the scalar products in $L^2(G \times G)$ and $\mathcal{B}_2(\mathcal{H})$, for proving relation (150) also in this case.

The proof of the theorem is complete. \square

Remark 5. In order to prove theorem 2, we could have shown that the map $\mathbf{U}: G \times G \rightarrow \mathcal{U}(\mathcal{B}_2(\mathcal{H}))$, defined by

$$\mathbf{U}(g_1, g_2) \hat{T} := U(g_1) \hat{T} U(g_2)^* =: \hat{T}(g_1, g_2), \quad g_1, g_2 \in G, \hat{T} \in \mathcal{B}_2(\mathcal{H}), \tag{166}$$

is an *irreducible* projective representation of the (unimodular) direct product group $G \times G$, and that, moreover, it is square integrable. Then, formula (150) can be regarded as the ‘orthogonality relations’ of the square integrable representation \mathbf{U} . The advantage of the above proof is that of ‘explicitly illustrating’ what happens for finite-rank operators. In the general case where G is not assumed to be unimodular—see remark 10 below—this kind of proof allows us to provide an explicit expression for (a variant of) the Duflo–Moore operator associated with the representation \mathbf{U} in terms of the Duflo–Moore operator associated with U .

Remark 6. Assume that the analyzing operator $\hat{T} \in \mathcal{B}_2(\mathcal{H})$ is a nonzero *finite-rank operator* ($\|\hat{T}\|_{\mathcal{B}_2(\mathcal{H})} = 1$). Then, arguing as in the proof of theorem 2, one shows that for every trace class operator $\hat{A} \in \mathcal{B}_1(\mathcal{H})$ and every bounded operator $\hat{B} \in \mathcal{B}(\mathcal{H})$ —setting: $B(g_1, g_2) := \text{tr}(\hat{T}(g_1, g_2)^* \hat{B})$ —the function

$$G \ni g_2 \mapsto (\mathfrak{D}_{\hat{T}} \hat{A})(g_1, g_2)^* B(g_1, g_2) \in \mathbb{C}, \quad \forall g_1 \in G, \tag{167}$$

is contained in $L^1(G)$, as well as the function $g_1 \mapsto \int_G (\mathfrak{D}_{\hat{T}} \hat{A})(g_1, g_2)^* B(g_1, g_2) d\mu_G(g_2)$, and the following formula holds:

$$\int_G d\mu_G(g_1) \int_G d\mu_G(g_2) (\mathfrak{D}_{\hat{T}} \hat{A})(g_1, g_2)^* B(g_1, g_2) d\mu_G \otimes \mu_G(g_1, g_2) = \text{tr}(\hat{A}^* \hat{B}), \tag{168}$$

where one can interchange the order of the integrals. Furthermore—taking into account the fact that, for any $\phi, \eta \in \mathcal{H}$, $\langle \phi, \hat{T}(g_1, g_2)\eta \rangle = (\mathfrak{D}_{\hat{T}}|\phi\rangle\langle\eta|)(g_1, g_2)^*$ —the following weak integral reconstruction formula holds:

$$\hat{B} = \int_G d\mu_G(g_1) \int_G d\mu_G(g_2) B(g_1, g_2) \hat{T}(g_1, g_2); \tag{169}$$

in particular, for $\hat{T} = |\psi\rangle\langle\psi|$, $\|\psi\| = 1$, we re-obtain relation (116).

Let us now investigate the intertwining property of the isometry $\mathfrak{D}_{\hat{T}}$ with respect to the natural action of the group G in $\mathcal{B}_2(\mathcal{H})$. Precisely, let us consider the representation $U \vee U : G \rightarrow \mathcal{U}(\mathcal{B}_2(\mathcal{H}))$ defined in section 5; see formula (130). As already observed, $U \vee U$ is a unitary representation, even in the case where the representation U is genuinely projective. Consider, now, the map $\mathcal{L}_M : G \rightarrow \mathcal{U}(\mathbb{L}^2(G \times G))$ defined by

$$(\mathcal{L}_M(g)f)(g_1, g_2) := M(g; g_1, g_2) f(g^{-1}g_1, g^{-1}g_2), \tag{170}$$

where the function $M : G \times G \times G \rightarrow \mathbb{T}$ is given by

$$M(g; g_1, g_2) := m(g^{-1}, g_1) m(g^{-1}, g_2)^*, \tag{171}$$

with m denoting the multiplier of U . The map \mathcal{L}_M is a unitary representation too, as the reader may verify by checking that the following relation holds:

$$M(gg'; g_1, g_2) = M(g; g_1, g_2) M(g'; g^{-1}g_1, g^{-1}g_2). \tag{172}$$

It is clear that the unitary representation \mathcal{L}_M is weakly Borel; hence, according to a well-known result, it is strongly continuous. Between the representations $U \vee U$ and \mathcal{L}_M there is a precise relation: $U \vee U$ is unitarily equivalent to a sub-representation of \mathcal{L}_M ; indeed, we have:

Proposition 8. *With the previous notations and assumptions, for every normalized Hilbert–Schmidt operator $\hat{T} \in \mathcal{B}_2(\mathcal{H})$, the isometry $\mathfrak{D}_{\hat{T}}$ intertwines the unitary representation $U \vee U : G \rightarrow \mathcal{U}(\mathcal{B}_2(\mathcal{H}))$ with the unitary representation $\mathcal{L}_M : G \rightarrow \mathcal{U}(\mathbb{L}^2(G \times G))$; namely:*

$$\mathfrak{D}_{\hat{T}} U \vee U(g) = \mathcal{L}_M(g) \mathfrak{D}_{\hat{T}}, \quad \forall g \in G. \tag{173}$$

Proof. Let \hat{A} an arbitrary operator in $\mathcal{B}_2(\mathcal{H})$. We want to prove that

$$(\mathfrak{D}_{\hat{T}}(U(g)\hat{A}U(g)^*))(g_1, g_2) = M(g; g_1, g_2) (\mathfrak{D}_{\hat{T}}\hat{A})(g^{-1}g_1, g^{-1}g_2). \tag{174}$$

In fact, the lhs of eq. (174) is equal to

$$\begin{aligned} \langle \hat{T}(g_1, g_2), U(g)\hat{A}U(g)^* \rangle_{\mathcal{B}_2(\mathcal{H})} &= \text{tr}(U(g_2)\hat{T}^*U(g_1)^*U(g)\hat{A}U(g)^*) \\ &= \text{tr}((U(g)^*U(g_2))\hat{T}^*(U(g)^*U(g_1))^*\hat{A}) \\ &= \text{tr}((m(g, g^{-1})U(g^{-1})U(g_2))\hat{T}^* \\ &\quad \times (m(g, g^{-1})U(g^{-1})U(g_1))^*\hat{A}) \\ &= \text{tr}((U(g^{-1})U(g_2))\hat{T}^*(U(g^{-1})U(g_1))^*\hat{A}). \end{aligned} \tag{175}$$

Hence, we have that

$$\begin{aligned} \langle \hat{T}(g_1, g_2), U(g)\hat{A}U(g)^* \rangle_{\mathcal{B}_2(\mathcal{H})} &= m(g^{-1}, g_1) m(g^{-1}, g_2)^* \text{tr}(U(g^{-1}g_2)\hat{T}^*U(g^{-1}g_1)^*\hat{A}) \\ &= M(g; g_1, g_2) \langle \hat{T}(g^{-1}g_1, g^{-1}g_2), \hat{A} \rangle_{\mathcal{B}_2(\mathcal{H})}. \end{aligned} \tag{176}$$

We have thus obtained the rhs of equation (174) and the proof is complete. \square

We conclude this section with a few remarks.

Remark 7. Let $\tilde{U} : G \rightarrow \mathcal{U}(\tilde{\mathcal{H}})$ be a projective representation physically equivalent to U (hence, square integrable too)

$$\tilde{U}(g) = \beta(g)WU(g)W^*, \quad \forall g \in G, \tag{177}$$

where $\beta: G \rightarrow \mathbb{T}$ is a Borel function and $W: \mathcal{H} \rightarrow \tilde{\mathcal{H}}$ a unitary or antiunitary operator. The unitary representations $U \vee U$ and $\tilde{U} \vee \tilde{U}$ are unitarily or antiunitarily equivalent (indeed, the operator $\mathcal{B}_2(\mathcal{H}) \ni \hat{A} \mapsto W\hat{A}W^* \in \mathcal{B}_2(\mathcal{H})$ is unitary if W is unitary, antiunitary if W is antiunitary). Moreover, denoting by \tilde{m} the multiplier of \tilde{U} and by $\tilde{M}: G \times G \times G \rightarrow \mathbb{T}$ the associated function defined as in (171), it turns out that the unitary representations $\mathcal{L}_{\tilde{M}}(g)$ and $\mathcal{L}_{\tilde{M}}(g)$ are, accordingly, unitarily or antiunitarily equivalent. Indeed—using the fact that for W unitary or antiunitary we have, respectively

$$\tilde{m}(g_1, g_2) = \frac{\beta(g_1g_2)}{\beta(g_1)\beta(g_2)}m(g_1, g_2) \quad \text{or} \quad \tilde{m}(g_1, g_2) = \frac{\beta(g_1g_2)}{\beta(g_1)\beta(g_2)}m(g_1, g_2)^* \tag{178}$$

one can easily check the following relations:

$$\tilde{M}(g; g_1, g_2) = \beta(g_1)^*\beta(g_2)\beta(g^{-1}g_1)\beta(g^{-1}g_2)^*M(g; g_1, g_2), \quad \text{for } W \text{ unitary}, \tag{179}$$

$$\tilde{M}(g; g_1, g_2) = \beta(g_1)^*\beta(g_2)\beta(g^{-1}g_1)\beta(g^{-1}g_2)^*M(g; g_1, g_2)^*, \quad \text{for } W \text{ antiunitary}. \tag{180}$$

Hence—denoting by J the standard complex conjugation in $L^2(G \times G)$, i.e. the antiunitary operator

$$J: L^2(G \times G) \ni f \mapsto f^* \in L^2(G \times G), \quad J = J^*, \tag{181}$$

and by $\hat{\beta}$ the multiplication operator in $L^2(G \times G)$ by the \mathbb{T} -valued Borel function $(g_1, g_2) \mapsto \beta(g_1)^*\beta(g_2)$ (operator which is obviously unitary)—for every $g \in G$ we have

$$\mathcal{L}_{\tilde{M}}(g) = \hat{\beta}\mathcal{L}_M(g)\hat{\beta}^*(W \text{ unitary}), \quad \mathcal{L}_{\tilde{M}}(g) = \hat{\beta}J\mathcal{L}_M(g)J\hat{\beta}^*(W \text{ antiunitary}). \tag{182}$$

This result is coherent with the fact that, denoting by $\tilde{\mathfrak{D}}_{\hat{T}'}$ the dequantization operator associated with the representation \tilde{U} , with analyzing operator $\hat{T}' \in \mathcal{B}_2(\tilde{\mathcal{H}})$ —where $\hat{T}' = W\hat{T}W^*$, for some $\hat{T} \in \mathcal{B}_2(\mathcal{H})$ such that $\|\hat{T}\|_{\mathcal{B}_2(\mathcal{H})} = 1$ —for every $\hat{A} \in \mathcal{B}_2(\mathcal{H})$ we have

$$\tilde{\mathfrak{D}}_{\hat{T}'}(W\hat{A}W^*) = (\hat{\beta}_W\mathfrak{D}_{\hat{T}})\hat{A}, \tag{183}$$

with $\hat{\beta}_W \equiv \hat{\beta}$, for W unitary, and $\hat{\beta}_W \equiv \hat{\beta}J$, for W antiunitary. We leave the simple check of relation (183) to the reader.

Remark 8. We stress that—excluding the trivial case where $\dim(\mathcal{H}) = 1$ — $\text{Ran}(\mathfrak{D}_{\hat{T}})$ is a proper subspace of $L^2(G \times G)$. In fact, if $\dim(\mathcal{H}) \geq 2$, according to relation (150) we have

$$\text{Ran}(\mathfrak{D}_{\hat{T}_1}) \perp \text{Ran}(\mathfrak{D}_{\hat{T}_2}), \quad \text{for all } \hat{T}_1, \hat{T}_2 \in \mathcal{B}_2(\mathcal{H}) \text{ such that } \langle \hat{T}_1, \hat{T}_2 \rangle_{\mathcal{B}_2(\mathcal{H})} = 0. \tag{184}$$

Hence, the ranges of a couple of dequantization maps, with mutually orthogonal analyzing operators, are mutually orthogonal subspaces of $L^2(G \times G)$. Therefore, the ranges of dequantization maps must be proper subspaces of $L^2(G \times G)$.

Remark 9. With every function $f \in L^2(G \times G)$ one can associate a function f° , contained in $L^2(G \times G)$ too, defined by

$$f^\circ(g_1, g_2) := f(g_2, g_1)^*, \quad \forall (g_1, g_2) \in G \times G. \tag{185}$$

Clearly, the antilinear application

$$\mathfrak{J}: L^2(G \times G) \ni f \mapsto f^\circ \in L^2(G \times G) \tag{186}$$

is a complex conjugation ($\mathfrak{J} = \mathfrak{J}^*$ and $\mathfrak{J}^2 = I$). Observe that, for every Hilbert–Schmidt operator $\hat{A} \in \mathcal{B}_2(\mathcal{H})$, the following relation holds:

$$\mathfrak{D}_{\hat{T}}(\hat{A}^*) = (\mathfrak{D}_{\hat{T}^*}\hat{A})^\diamond. \tag{187}$$

Indeed, we have

$$\begin{aligned} \mathfrak{D}_{\hat{T}}(\hat{A}^*)(g_1, g_2) &= \text{tr}(U(g_2)\hat{T}^*U(g_1)^*\hat{A}^*) = \text{tr}(\hat{A}U(g_1)\hat{T}U(g_2)^*)^* \\ &= \text{tr}(U(g_1)\hat{T}U(g_2)^*\hat{A})^* = (\mathfrak{D}_{\hat{T}^*}\hat{A})^\diamond(g_1, g_2). \end{aligned} \tag{188}$$

Suppose that the analyzing operator $\hat{T} \in \mathcal{B}_2(\mathcal{H})$ is selfadjoint. Then, the isometry $\mathfrak{D}_{\hat{T}}$ intertwines the standard complex conjugation $\hat{A} \mapsto \hat{A}^*$ in $\mathcal{B}_2(\mathcal{H})$ with the complex conjugation \mathfrak{J} in $L^2(G \times G)$, i.e. $\mathfrak{D}_{\hat{T}}(\hat{A}^*) = (\mathfrak{D}_{\hat{T}}\hat{A})^\diamond$. Therefore, taking into account the injectivity of the map $\mathfrak{D}_{\hat{T}}$, a function Ψ belonging to $\text{Ran}(\mathfrak{D}_{\hat{T}})$ is the image of a selfadjoint operator *if and only if* $\Psi = \Psi^\diamond$.

Remark 10. Up to this point, we have focused on the case where the group G is unimodular. We stress that a suitable dequantization map can be defined even if G is *not* unimodular (we denote by μ_G , as usual, a left Haar measure on G and by \hat{D}_U the Duflo–Moore operator normalized according to μ_G), though in this case the construction is slightly more complicated. Here we will sketch the main points of this construction; further details (and suitable examples) will be contained in a forthcoming paper. Let us denote by $\text{FR}(\mathcal{H})$ the linear span of finite-rank operators and let us consider the set

$$\check{\text{FR}}(\mathcal{H}; U) := \{\hat{F} \in \text{FR}(\mathcal{H}) : \text{Ran}(\hat{F}), \text{Ran}(\hat{F}^*) \subset \text{Dom}(\hat{D}_U)\}. \tag{189}$$

The set $\check{\text{FR}}(\mathcal{H}; U)$ is a dense linear span in $\mathcal{B}_2(\mathcal{H})$, and a generic nonzero vector in $\check{\text{FR}}(\mathcal{H}; U)$ is of the form $\sum_{n=1}^N |\psi_n\rangle\langle\phi_n|$, where $\{\psi_n\}_{n=1}^N, \{\phi_n\}_{n=1}^N$ are linearly independent sets in $\text{Dom}(\hat{D}_U)$. Let us introduce a linear operator $\check{\mathfrak{K}}_U$, with domain $\check{\text{FR}}(\mathcal{H}; U)$, defined by

$$\check{\mathfrak{K}}_U \left(\sum_{n=1}^N |\psi_n\rangle\langle\phi_n| \right) = \sum_{n=1}^N |\hat{D}_U\psi_n\rangle\langle\hat{D}_U\phi_n|. \tag{190}$$

It is easy to check that, due to the selfadjointness of \hat{D}_U , $\check{\mathfrak{K}}_U$ is a symmetric operator. It follows that $\check{\mathfrak{K}}_U$ is closable, and we denote by \mathfrak{K}_U the closure of $\check{\mathfrak{K}}_U$; hence, \mathfrak{K}_U is a closed, symmetric, densely defined operator in $\mathcal{B}_2(\mathcal{H})$ whose restriction to $\check{\text{FR}}(\mathcal{H}; U)$ coincides with $\check{\mathfrak{K}}_U$.

At this point, with every operator \hat{T} in the dense linear span $\text{Dom}(\mathfrak{K}_U)$ one can associate a linear map $\mathfrak{D}_{\hat{T}} : \mathcal{B}_2(\mathcal{H}) \rightarrow L^2(G \times G) \equiv L^2(G \times G, \mu_G \otimes \mu_G; \mathbb{C})$ defined by

$$(\mathfrak{D}_{\hat{T}}\hat{A})(g_1, g_2) := \langle U(g_1)\hat{T}U(g_2)^*, \hat{A} \rangle_{\mathcal{B}_2(\mathcal{H})}, \quad g_1, g_2 \in G, \tag{191}$$

which—for \hat{T} nonzero and such that $\|\mathfrak{K}_U\hat{T}\|_{\mathcal{B}_2(\mathcal{H})} = 1$ —is an isometry. Moreover, for any $\hat{A}, \hat{B} \in \mathcal{B}_2(\mathcal{H})$ and any \hat{S}, \hat{T} in the dense linear span $\text{Dom}(\mathfrak{K}_U) \subset \mathcal{B}_2(\mathcal{H})$, the following orthogonality relations hold:

$$\langle \mathfrak{D}_{\hat{T}}\hat{A}, \mathfrak{D}_{\hat{S}}\hat{B} \rangle_{L^2(G \times G)} = \langle \hat{A}, \hat{B} \rangle_{\mathcal{B}_2(\mathcal{H})} \langle \mathfrak{K}_U\hat{S}, \mathfrak{K}_U\hat{T} \rangle_{\mathcal{B}_2(\mathcal{H})}. \tag{192}$$

The proof of these statements goes along lines similar to the ones traced in the proof of theorem 2. First one proves the statements with the operator \hat{T} (and \hat{S}) belonging to the dense linear span $\check{\text{FR}}(\mathcal{H}; U)$. Then, one extends the result to a generic \hat{T} in $\text{Dom}(\mathfrak{K}_U)$ by means of a limit argument. This time the sequence $\{\hat{T}_N\}_{N \in \mathbb{N}}$ converging to \hat{T} should be chosen as follows. It must be a sequence in $\check{\text{FR}}(\mathcal{H}; U)$ such that

$$\lim_{N \rightarrow \infty} \|\hat{T} - \hat{T}_N\|_{\mathcal{B}_2(\mathcal{H})} = 0 \quad \text{and} \quad \lim_{N \rightarrow \infty} \|\mathfrak{K}_U\hat{T} - \mathfrak{K}_U\hat{T}_N\|_{\mathcal{B}_2(\mathcal{H})} = 0 \tag{193}$$

(such a sequence exists since \mathfrak{K}_U is the closure of $\check{\mathfrak{K}}_U$). One can prove that the operator $\check{\mathfrak{K}}_U$ is essentially selfadjoint; hence, its closure \mathfrak{K}_U is the unique selfadjoint extension of $\check{\mathfrak{K}}_U$. Thus, \mathfrak{K}_U is a variant (remark 4) of the Duflo–Moore operator associated with the square integrable projective representation \mathbf{U} , see remark 5. Therefore, for $\hat{T} \in \text{Dom}(\mathfrak{K}_U)$ such that $\|\mathfrak{K}_U \hat{T}\|_{\mathcal{B}_2(\mathcal{H})} = 1$, the linear map $\mathfrak{D}_{\hat{T}}$ can be regarded as the generalized wavelet transform generated by \mathbf{U} , with analyzing vector \hat{T} .

In the following section, we will exploit the class of frames introduced above and the results of section 2 for deriving suitable expressions of quantum-mechanical formulae in terms of functions on ‘phase space’. Although most of the results hold in the general case, we will assume, for the sake of simplicity, that the l.c.s.c. group G is unimodular.

7. Frame transforms and quantum mechanics

Since we are now equipped with a wide class of tight frames in the space $\mathcal{B}_2(\mathcal{H})$ of Hilbert–Schmidt operators in the Hilbert space \mathcal{H} , we can exploit the results of section 2. It will be convenient to denote by \mathbf{G} the direct product group $G \times G$ (G unimodular), by $\mathbf{g} \equiv (g_1, g_2)$ a typical element of \mathbf{G} , by $\widehat{\mathbf{g}}$ the ‘diagonal element’ (g, g) of \mathbf{G} and by μ_G the Haar measure $\mu_G \otimes \mu_G$ on \mathbf{G} (which is, obviously, a unimodular l.c.s.c. group). Then, according to theorem 2, for every nonzero Hilbert–Schmidt operator $\hat{T} \in \mathcal{B}_2(\mathcal{H})$ such that $\|\hat{T}\|_{\mathcal{B}_2(\mathcal{H})} = 1$ (μ_G is normalized in agreement with U), the family of operators

$$\{\hat{T}(\mathbf{g}) \equiv U(g_1)\hat{T}U(g_2)^* = \mathbf{U}(\mathbf{g})\hat{T}\}_{\mathbf{g} \in \mathbf{G}}, \tag{194}$$

is a normalized tight frame in $\mathcal{B}_2(\mathcal{H})$, based on (\mathbf{G}, μ_G) . Thus, we can identify the measure space (Y, ν) of section 2 with the measure space (\mathbf{G}, μ_G) . The frame transform associated with the frame (194) is the linear map $\mathfrak{D}_{\hat{T}}: \mathcal{B}_2(\mathcal{H}) \rightarrow L^2(\mathbf{G}) \equiv L^2(\mathbf{G}, \mu_G; \mathbb{C})$ — $(\mathfrak{D}_{\hat{T}} \hat{A})(\mathbf{g}) := \langle \hat{T}(\mathbf{g}), \hat{A} \rangle_{\mathcal{B}_2(\mathcal{H})}$, for every $\hat{A} \in \mathcal{B}_2(\mathcal{H})$ —which is an isometry (the ‘dequantization map’). We will denote by $\mathfrak{Q}_{\hat{T}}: L^2(\mathbf{G}) \rightarrow \mathcal{B}_2(\mathcal{H})$ the adjoint of the isometry $\mathfrak{D}_{\hat{T}}$; then, $\mathfrak{Q}_{\hat{T}}$ (the ‘quantization map’) is a partial isometry that coincides with the pseudo-inverse of $\mathfrak{D}_{\hat{T}}$

$$\mathfrak{Q}_{\hat{T}} \mathfrak{D}_{\hat{T}} = I, \quad \text{Ker}(\mathfrak{Q}_{\hat{T}}) = \text{Ran}(\mathfrak{D}_{\hat{T}})^\perp. \tag{195}$$

For the partial isometry $\mathfrak{Q}_{\hat{T}}$ we have the following simple formula (compare with relation (8)):

$$\mathfrak{Q}_{\hat{T}} \Phi = \int_G d\mu_G(g) \Phi(g) \hat{T}(g), \quad \forall \Phi \in L^2(\mathbf{G}). \tag{196}$$

We stress that the integral in formula (196) is a weak integral of $\mathcal{B}_2(\mathcal{H})$ -valued functions; hence, *a fortiori*, it can also be regarded as a weak integral of bounded-operator-valued functions (see remark 2).

As observed in section 2, the linear maps $\mathfrak{D}_{\hat{T}}$ and $\mathfrak{Q}_{\hat{T}}$ induce in $L^2(\mathbf{G})$ a star product of functions defined by (see definition (38))

$$\Phi_1 \star \Phi_2 := \mathfrak{D}_{\hat{T}}((\mathfrak{Q}_{\hat{T}} \Phi_1)(\mathfrak{Q}_{\hat{T}} \Phi_2)), \quad \forall \Phi_1, \Phi_2 \in L^2(\mathbf{G}). \tag{197}$$

According to proposition 3, we have

$$(\Phi_1 \star \Phi_2)(\mathbf{g}) = \int_G d\mu_G(g') \int_G d\mu_G(g'') \kappa_{\hat{T}}(\mathbf{g}, g', g'') \Phi_1(g') \Phi_2(g''), \tag{198}$$

where

$$\kappa_{\hat{T}}(\mathbf{g}, g', g'') := \langle \hat{T}(\mathbf{g}), \hat{T}(g') \hat{T}(g'') \rangle_{\mathcal{B}_2(\mathcal{H})} = \text{tr}(\hat{T}(\mathbf{g})^* \hat{T}(g') \hat{T}(g'')). \tag{199}$$

In particular, the subspace $\text{Ran}(\mathfrak{D}_{\hat{T}})$ of $L^2(G)$ is a r.k.H.s. (compare with formulae (17) and (18))

$$\Phi(g) = \int_G d\mu_G(g') \kappa_{\hat{T}}(g, g') \Phi(g'), \quad \forall \Phi \in \text{Ran}(\mathfrak{D}_{\hat{T}}), \quad (200)$$

where the reproducing kernel has the following expression:

$$\kappa_{\hat{T}}(g, g') := \langle \hat{T}(g), \hat{T}(g') \rangle_{\mathcal{B}_2(\mathcal{H})}, \quad (201)$$

and, for every couple of Hilbert–Schmidt operators $\hat{A}_1, \hat{A}_2 \in \mathcal{B}_2(\mathcal{H})$, we have

$$(\mathfrak{D}_{\hat{T}} \hat{A}_1 \hat{A}_2)(g) = \int_G d\mu_G(g') \int_G d\mu_G(g'') \kappa_{\hat{T}}(g, g', g'') A_1(g') A_2(g''), \quad (202)$$

with $A_1(g) \equiv (\mathfrak{D}_{\hat{T}} \hat{A}_1)(g)$, $A_2(g) \equiv (\mathfrak{D}_{\hat{T}} \hat{A}_2)(g)$.

Observe that it is possible to express, within the present framework, the expectation values of quantum-mechanical observables. Recall, in fact, that the (bounded) left and right multiplication operators in $\mathcal{B}_2(\mathcal{H})$ by a bounded operator \hat{A} —i.e., respectively, the linear operators: $L_{\hat{A}}: \mathcal{B}_2(\mathcal{H}) \ni \hat{B} \mapsto \hat{A}\hat{B} \in \mathcal{B}_2(\mathcal{H})$ and $R_{\hat{A}}: \mathcal{B}_2(\mathcal{H}) \ni \hat{B} \mapsto \hat{B}\hat{A} \in \mathcal{B}_2(\mathcal{H})$ —are represented as suitable integral operators in the Hilbert space of frame transforms $\text{Ran}(\mathfrak{D}_{\hat{T}}) = \mathfrak{D}_{\hat{T}}(\mathcal{B}_2(\mathcal{H}))$. Precisely, the ‘left’ and ‘right’ integral kernels

$$\chi_{\hat{T}}^L(\hat{A}; g, g') := \langle \hat{T}(g), \hat{A}\hat{T}(g') \rangle_{\mathcal{B}_2(\mathcal{H})}, \quad \chi_{\hat{T}}^R(\hat{A}; g, g') := \langle \hat{T}(g), \hat{T}(g')\hat{A} \rangle_{\mathcal{B}_2(\mathcal{H})} \quad (203)$$

see proposition 4—correspond to the ‘super-operators’ $L_{\hat{A}}$ and $R_{\hat{A}}$, respectively. In particular, for every trace class operator $\hat{\rho} \in \mathcal{B}_1(\mathcal{H})$, the following formulae apply:

$$(\mathfrak{D}_{\hat{T}} \hat{A} \hat{\rho})(g) = \int_G d\mu_G(g') \chi_{\hat{T}}^L(\hat{A}; g, g') \rho(g'), \quad \rho \equiv \mathfrak{D}_{\hat{T}} \hat{\rho}, \quad (204)$$

$$(\mathfrak{D}_{\hat{T}} \hat{\rho} \hat{A})(g) = \int_G d\mu_G(g') \chi_{\hat{T}}^R(\hat{A}; g, g') \rho(g'). \quad (205)$$

Besides, for every normalized nonzero vector ψ in \mathcal{H} —more precisely, for every rank one projector $\hat{\psi} \equiv |\psi\rangle\langle\psi|$ —setting

$$\gamma_{\hat{T}, \hat{\psi}}(g, g) := \langle U \vee U(g) \hat{\psi}, \hat{T}(g) \rangle_{\mathcal{B}_2(\mathcal{H})} = \langle U(g) \psi, \hat{T}(g) U(g) \psi \rangle, \quad (206)$$

we have (see proposition 5; consider that $\{U(g)\psi\}_{g \in G}$ is a normalized tight frame in \mathcal{H} , based on (G, μ_G))

$$\text{tr}(\hat{\rho}) = \int_G d\mu_G(g) \int_G d\mu_G(g) \gamma_{\hat{T}, \hat{\psi}}(g, g) \rho(g) \equiv \text{tr}(\rho). \quad (207)$$

According to the second assertion of proposition 5, a *positive* Hilbert–Schmidt operator $\hat{B} \in \mathcal{B}_2(\mathcal{H})$ is a trace class operator if and only if

$$\int_G d\mu_G(g) \int_G d\mu_G(g) \gamma_{\hat{T}, \hat{\psi}}(g, g) (\mathfrak{D}_{\hat{T}} \hat{B})(g) < +\infty. \quad (208)$$

Observe also that, recalling the intertwining relation (173), from definition (206) we get

$$\begin{aligned} \gamma_{\hat{T}, \hat{\psi}}(g, g) &:= \langle U \vee U(g) \hat{\psi}, \hat{T}(g) \rangle_{\mathcal{B}_2(\mathcal{H})} = \langle \hat{T}(g), U \vee U(g) \hat{\psi} \rangle_{\mathcal{B}_2(\mathcal{H})}^* \\ &= (\mathfrak{D}_{\hat{T}} U \vee U(g) \hat{\psi})(g)^* \\ &= (\mathcal{L}_M(g) \mathfrak{D}_{\hat{T}} \hat{\psi})(g)^*. \end{aligned} \quad (209)$$

Remark 11. Formula (207) is a special case of a more general relation. In fact, let \hat{S} be a trace class operator in \mathcal{H} such that $\text{tr}(\hat{S}) = 1$; then, extending definition (206), let us set

$$\gamma_{\hat{T}, \hat{S}}(g, g) := \langle U \vee U(g) \hat{S}, \hat{T}(g) \rangle_{\mathcal{B}_2(\mathcal{H})} = \text{tr}((U \vee U(g) \hat{S}^*) \hat{T}(g)). \quad (210)$$

At this point, using the ‘second trace formula’ (119) and the reconstruction formula for the operator $\hat{\rho}$, we find

$$\begin{aligned} \text{tr}(\hat{\rho}) &= \int_G d\mu_G(g) \text{tr}((U \vee U(g)\hat{S}^*)\hat{\rho}) \\ &= \int_G d\mu_G(g) \int_G d\mu_G(\mathbf{g}) \gamma_{\hat{T}, \hat{S}}(g, \mathbf{g}) \rho(\mathbf{g}). \end{aligned} \tag{211}$$

Moreover, arguing as above, we conclude that

$$\gamma_{\hat{T}, \hat{S}}(g, \mathbf{g}) = (\mathcal{L}_M(g)\mathfrak{D}_{\hat{T}}\hat{S})(\mathbf{g})^*. \tag{212}$$

This formula shows that the function $\mathbf{g} \mapsto \gamma_{\hat{T}, \hat{S}}(g, \mathbf{g})^*$ is contained in $\text{Ran}(\mathfrak{D}_{\hat{T}})$.

In the special case where $\hat{T} \in \mathcal{B}_1(\mathcal{H})$, exploiting again the second trace formula (119), we find also that

$$\text{tr}(\hat{\rho}) \text{tr}(\hat{T})^* = \text{tr}(\hat{\rho}) \text{tr}(\hat{T}^*) = \int_G d\mu_G(g) \rho(\widehat{g}), \quad \rho \equiv \mathfrak{D}_{\hat{T}}\hat{\rho}. \tag{213}$$

Hence, in particular, $|\text{tr}(\hat{T})|^2 = \int_G d\mu_G(g) (\mathfrak{D}_{\hat{T}}\hat{T})(\widehat{g})$, and, if $\hat{T} \in \mathcal{B}_1(\mathcal{H})$ is such that $\text{tr}(\hat{T}) \neq 0$, we have

$$|\text{tr}(\hat{\rho})| = \frac{1}{\sqrt{\int_G d\mu_G(g) (\mathfrak{D}_{\hat{T}}\hat{T})(\widehat{g})}} \left| \int_G d\mu_G(g) \rho(\widehat{g}) \right|. \tag{214}$$

We are now ready to provide a suitable expression for the quantity $\text{tr}(\hat{A}\hat{\rho})$, which—in the special case where the bounded operator \hat{A} is selfadjoint, and the trace class operator $\hat{\rho}$ is positive and of unit trace—can be regarded as a quantum-mechanical expectation value. From relations (204), (205) and (207) it follows immediately that

$$\begin{aligned} \text{tr}(\hat{A}\hat{\rho}) &= \int_G d\mu_G(g) \int_G d\mu_G(\mathbf{g}) \int_G d\mu_G(\mathbf{g}') \gamma_{\hat{T}, \hat{\psi}}(g, \mathbf{g}) \chi_{\hat{T}}^L(\hat{A}; \mathbf{g}, \mathbf{g}') \rho(\mathbf{g}') \\ &= \int_G d\mu_G(g) \int_G d\mu_G(\mathbf{g}) \int_G d\mu_G(\mathbf{g}') \gamma_{\hat{T}, \hat{\psi}}(g, \mathbf{g}) \chi_{\hat{T}}^R(\hat{A}; \mathbf{g}, \mathbf{g}') \rho(\mathbf{g}'). \end{aligned} \tag{215}$$

Of course, analogous formulae involving the more general type of integral kernel $\gamma_{\hat{T}, \hat{S}}(\cdot, \cdot)$ defined above hold too. Moreover, in the special case where $\hat{T} \in \mathcal{B}_1(\mathcal{H})$, with $\text{tr}(\hat{T}) \neq 0$, formula (213) implies

$$\begin{aligned} \text{tr}(\hat{A}\hat{\rho}) &= \text{tr}(\hat{T}^*)^{-1} \int_G d\mu_G(g) \int_G d\mu_G(\mathbf{g}') \chi_{\hat{T}}^L(\hat{A}; \widehat{g}, \mathbf{g}') \rho(\mathbf{g}') \\ &= \text{tr}(\hat{T}^*)^{-1} \int_G d\mu_G(g) \int_G d\mu_G(\mathbf{g}') \chi_{\hat{T}}^R(\hat{A}; \widehat{g}, \mathbf{g}') \rho(\mathbf{g}') = \text{tr}(\hat{\rho}\hat{A}). \end{aligned} \tag{216}$$

In conclusion, having in mind applications to quantum mechanics, within the framework outlined in the present section we have the following picture. With *states* (density operators) are associated functions—the frame transforms of the density operators—belonging to the r.k.H.s. $\text{Ran}(\mathfrak{D}_{\hat{T}})$, which is endowed with a star product that reproduces the product of the H^* -algebra $\mathcal{B}_2(\mathcal{H})$. On the other hand, with *observables* are associated suitable (left and right) integral kernels. The quantum-mechanical expectation values are given by integral formulae involving the frame transforms associated with states and the integral kernels. Note that in this picture the norm of a quantum observable can be defined ‘intrinsically’. Indeed, for every

bounded selfadjoint operator \hat{A} in \mathcal{H} , recalling definition (42) and relation (44), and using the fact that $L_{\hat{A}}$ is a bounded *selfadjoint* operator in $\mathcal{B}_2(\mathcal{H})$, we have

$$\begin{aligned} \|\hat{A}\| &= \|L_{\hat{A}}\| \\ &= \sup_{0 \neq \hat{B} \in \mathcal{B}_2(\mathcal{H})} \|\hat{B}\|_{\mathcal{B}_2(\mathcal{H})}^{-2} |\langle \hat{B}, \hat{A}\hat{B} \rangle_{\mathcal{B}_2(\mathcal{H})}| \quad (L_{\hat{A}} \text{ selfadjoint}) \\ &= \sup_{0 \neq \Phi \in \text{Ran}(\mathfrak{D}_{\hat{T}})} \|\Phi\|_{L^2(\mathcal{G})}^{-2} \left| \int_{\mathcal{G}} d\mu_G(g) \int_{\mathcal{G}} d\mu_G(g') \chi_{\hat{T}}^L(\hat{A}; g, g') \Phi(g)^* \Phi(g') \right|. \end{aligned} \quad (217)$$

Moreover, taking into account relation (56), we find out that in formula (217) one can relax the condition that $\Phi \in \text{Ran}(\mathfrak{D}_{\hat{T}})$; i.e.

$$\begin{aligned} \|\hat{A}\| &= \sup_{0 \neq \Phi \in L^2(\mathcal{G})} \|\Phi\|_{L^2(\mathcal{G})}^{-2} \left| \int_{\mathcal{G}} d\mu_G(g) \int_{\mathcal{G}} d\mu_G(g') \chi_{\hat{T}}^L(\hat{A}; g, g') \Phi(g)^* \Phi(g') \right| \\ &=: \|\chi_{\hat{T}}^L(\hat{A}; \cdot, \cdot)\|. \end{aligned} \quad (218)$$

Of course, using the fact that $\|\hat{A}\| = \|R_{\hat{A}}\|$, one obtains a completely analogous relation involving the right integral kernel $\chi_{\hat{T}}^R(\hat{A}; \cdot, \cdot)$.

Therefore, we can identify the Jordan–Lie algebra of bounded selfadjoint operators in \mathcal{H} with the vector space of the associated left integral kernels endowed with the norm defined by formula (218), and with the Jordan product and the Lie bracket defined by (compare with formulae (60) and (61), respectively)

$$\begin{aligned} \chi_{\hat{T}}^L(\hat{A}_1 \circ \hat{A}_2; g', g'') &= \frac{1}{2} \int_{\mathcal{G}} d\mu_G(g) (\chi_{\hat{T}}^L(\hat{A}_1; g', g) \chi_{\hat{T}}^L(\hat{A}_2; g, g'')) \\ &\quad + \chi_{\hat{T}}^L(\hat{A}_2; g', g) \chi_{\hat{T}}^L(\hat{A}_1; g, g'') =: \chi_{\hat{T}}^L(\hat{A}_1; \cdot, \cdot) \circ \chi_{\hat{T}}^L(\hat{A}_2; \cdot, \cdot), \end{aligned} \quad (219)$$

$$\begin{aligned} \chi_{\hat{T}}^L(\{\hat{A}_1, \hat{A}_2\}; g', g'') &= \frac{1}{i} \int_{\mathcal{G}} d\mu_G(g) (\chi_{\hat{T}}^L(\hat{A}_1; g', g) \chi_{\hat{T}}^L(\hat{A}_2; g, g'') \\ &\quad - \chi_{\hat{T}}^L(\hat{A}_2; g', g) \chi_{\hat{T}}^L(\hat{A}_1; g, g'')) =: \{\chi_{\hat{T}}^L(\hat{A}_1; \cdot, \cdot), \chi_{\hat{T}}^L(\hat{A}_2; \cdot, \cdot)\}, \end{aligned} \quad (220)$$

for any couple of bounded selfadjoint operators $\hat{A}_1, \hat{A}_2 \in \mathcal{B}(\mathcal{H})$. It is clear that a similar identification holds for the (suitably equipped) vector space of right integral kernels.

Assume now that the analyzing operator $\hat{T} \in \mathcal{B}_2(\mathcal{H})$ is selfadjoint. Observe that, in this case, the image through $\mathfrak{D}_{\hat{T}}$ of the set $\mathcal{P}(\mathcal{H})$ of *pure states* (rank-one projectors) in the Hilbert space \mathcal{H} is characterized as a subset of $\text{Ran}(\mathfrak{D}_{\hat{T}})$ in the following way:

$$\mathfrak{D}_{\hat{T}}(\mathcal{P}(\mathcal{H})) = \{\Psi \in \text{Ran}(\mathfrak{D}_{\hat{T}}) : \Psi = \Psi^\diamond, \Psi \star \Psi = \Psi, \text{tr}(\Psi) = 1\}, \quad (221)$$

where

$$\text{tr}(\Psi) = \int_{\mathcal{G}} d\mu_G(g) \int_{\mathcal{G}} d\mu_G(g) \gamma_{\hat{T}, \hat{\psi}}(g, g) \Psi(g). \quad (222)$$

Indeed—recalling remark 9, and formulae (202) and (207)—the image through the isometry $\mathfrak{D}_{\hat{T}}$ of the set of orthogonal projectors in \mathcal{H} is characterized by the couple of conditions

$$\Psi = \Psi^\diamond, \quad \Psi \star \Psi = \Psi. \quad (223)$$

At this point, the third condition— $\text{tr}(\Psi) = 1$ —ensures that $\mathfrak{D}_{\hat{T}}\Psi$ is a trace class operator (note that $\mathfrak{D}_{\hat{T}}\Psi$ is positive and recall condition (208)), i.e. a finite-rank projector, and in particular a rank one projector. This characterization of the set $\mathfrak{D}_{\hat{T}}(\mathcal{P}(\mathcal{H}))$ allows us to obtain an alternative expression of the norm of an observable in terms of its left and right integral kernels. In fact, for every bounded selfadjoint operator \hat{A} in \mathcal{H} , we have that

$$\|\hat{A}\| = \sup_{\psi \in \mathcal{H} : \|\psi\|=1} |\langle \psi, \hat{A}\psi \rangle| = \sup_{\hat{P} \in \mathcal{P}(\mathcal{H})} |\text{tr}(\hat{A}\hat{P})|. \quad (224)$$

Therefore, if the analyzing operator $\hat{T} \in \mathcal{B}_2(\mathcal{H})$ is selfadjoint, in terms of the left integral kernel $\chi_{\hat{T}}^L(\hat{A}; \cdot, \cdot)$, the norm of the operator \hat{A} has the following alternative expression:

$$\|\hat{A}\| = \sup \left\{ \left\| \int_G d\mu_G(g) \int_G d\mu_G(g') \int_G d\mu_G(g'') \gamma_{\hat{T}, \hat{\Psi}}(g, g', g'') \chi_{\hat{T}}^L(\hat{A}; g, g'') \Psi(g'') \right\| : \Psi \in \text{Ran}(\mathfrak{D}), \Psi = \Psi^\circ, \Psi \star \Psi = \Psi, \text{tr}(\Psi) = 1 \right\} = \|\chi_{\hat{T}}^L(\hat{A}; \cdot, \cdot)\|. \quad (225)$$

Clearly, an analogous expression involving the right integral kernel $\chi_{\hat{T}}^R(\hat{A}; \cdot, \cdot)$ holds too.

We leave to the reader the simple exercise of deriving how the natural symmetry action of the group G on bounded operators in \mathcal{H} is represented in the vector spaces of the associated left and right integral kernels.

8. A remarkable example

In this section we will focus on the case where the group G is the additive group $\mathbb{R} \times \mathbb{R}$ (the group of translations on the 1 + 1-dimensional phase space; the generalization to the $n + n$ -dimensional case is straightforward) and the square integrable projective representation U is the Weyl system (137). We will denote a generic element of $G \equiv \mathbb{R} \times \mathbb{R}$ as a complex variable— $z \equiv q + ip$ —and a generic element of the direct product group $\mathcal{G} \equiv (\mathbb{R} \times \mathbb{R}) \times (\mathbb{R} \times \mathbb{R})$, accordingly, as $z = (z_1, z_2)$. As in section 7, the diagonal element (z, z) of \mathcal{G} will be denoted by \hat{z} . We recall that the Haar measure μ_G on $G \equiv \mathbb{R} \times \mathbb{R}$, normalized in agreement with U , is given by $d\mu_G(z) = (2\pi)^{-1} dz \equiv (2\pi)^{-1} dq dp$; hence, the Haar measure μ_G on \mathcal{G} is given by $d\mu_{\mathcal{G}}(z) = (2\pi)^{-2} dz \equiv (2\pi)^{-2} dz_1 dz_2$. At this point, as a consequence of theorem 2, we have that for every normalized nonzero Hilbert–Schmidt operator \hat{T} in $L^2(\mathbb{R})$ the family of operators

$$\{\hat{T}(z) \equiv U(z_1)\hat{T}U(z_2)^* = \mathbf{U}(z)\hat{T}\}_{z \in G} \quad (226)$$

is a normalized tight frame in $\mathcal{B}_2(L^2(\mathbb{R}))$, based on $(\mathcal{G}, \mu_{\mathcal{G}})$. This frame allows us to define the isometry

$$\mathfrak{D}_{\hat{T}}: \mathfrak{B}_2 \equiv \mathcal{B}_2(L^2(\mathbb{R})) \rightarrow \mathfrak{L}^2 \equiv L^2(\mathcal{G}) \quad (227)$$

by setting

$$(\mathfrak{D}_{\hat{T}}\hat{A})(z) := \langle \hat{T}(z), \hat{A} \rangle_{\mathfrak{B}_2}, \quad \forall \hat{A} \in \mathfrak{B}_2. \quad (228)$$

The range of the isometry $\mathfrak{D}_{\hat{T}}$ is a proper subspace of \mathfrak{L}^2 and a r.k.H.s. (embedded in \mathfrak{L}^2), with reproducing kernel

$$\kappa_{\hat{T}}(z, \tilde{z}) := \langle \hat{T}(z), \hat{T}(\tilde{z}) \rangle_{\mathfrak{B}_2}; \quad (229)$$

taking into account the fact that $U(z)^* = U(-z)$, we have

$$\begin{aligned} \kappa_{\hat{T}}(z, \tilde{z}) &= \text{tr}(U(z_2)\hat{T}^*U(-z_1)U(\tilde{z}_1)\hat{T}U(-\tilde{z}_2)) \\ &= e^{\frac{i}{4}(z_1^*\tilde{z}_1 - z_1\tilde{z}_1^*)} e^{-\frac{i}{4}(z_2^*\tilde{z}_2 - z_2\tilde{z}_2^*)} \text{tr}(U(z_2 - \tilde{z}_2)\hat{T}^*U(z_1 - \tilde{z}_1)^*\hat{T}) \\ &= \exp\left(\frac{i}{4}(z_1^*\tilde{z}_1 - z_1\tilde{z}_1^* - z_2^*\tilde{z}_2 + z_2\tilde{z}_2^*)\right) (\mathfrak{D}_{\hat{T}}\hat{T})(z - \tilde{z}), \end{aligned} \quad (230)$$

with $z \equiv (z_1, z_2)$, $\tilde{z} \equiv (\tilde{z}_1, \tilde{z}_2)$. Moreover, the isometry $\mathfrak{D}_{\hat{T}}$ intertwines the unitary representation $U \vee U: G \equiv \mathbb{R} \times \mathbb{R} \rightarrow \mathcal{U}(\mathfrak{B}_2)$,

$$U \vee U(z)\hat{A} = U(z)\hat{A}U(-z), \quad U(-z) = U(z)^*, \quad (231)$$

with the unitary representation $\mathcal{L}_{\mathfrak{M}}: G \rightarrow \mathcal{U}(\mathfrak{L}^2)$ defined by

$$(\mathcal{L}_{\mathfrak{M}}(z)f)(z) := \mathbf{M}(z; z)f(z - \hat{z}), \quad \forall f \in \mathfrak{L}^2, \quad (232)$$

where

$$M(z; z) := \exp\left(\frac{i}{2}(q(p_2 - p_1) - p(q_2 - q_1))\right), \quad z \equiv q + ip, \quad z \equiv (q_1 + ip_1, q_2 + ip_2). \tag{233}$$

Of course all the formulae obtained in section 7 apply to this case; we will present some detailed calculations and examples elsewhere. We want now to highlight, briefly, the relation between our results and the fundamental seminal papers [18] of Cahill and Glauber on quasi-distributions. In the cited papers, Cahill and Glauber (with aims partially distinct from ours) introduced and studied a family of normal operators with spectral decomposition

$$\hat{T}_s := \frac{2}{1-s} \sum_{n=0}^{\infty} \left(\frac{s+1}{s-1}\right)^n |n\rangle\langle n|, \quad s \in \mathbb{C}, s \neq 1, \tag{234}$$

where $\{|n\rangle\}_{n=0,1,\dots}$ are the standard eigenfunctions of the harmonic oscillator Hamiltonian. From the first of the papers [18] we learn, in particular, the following (easily verifiable) facts⁸.

- For $\text{Re}(s) \leq 0$, the operator \hat{T}_s is bounded and

$$\|\hat{T}_s\| = \left| \frac{2}{1-s} \right|; \tag{235}$$

moreover, $\hat{T}_s^* = \hat{T}_{s^*}$.

- For $\text{Re}(s) < 0$, the operator \hat{T}_s belongs to the Banach space $\mathcal{B}_1(L^2(\mathbb{R}))$ (hence, in particular, to the Hilbert space $\mathfrak{B}_2 \equiv \mathcal{B}_2(L^2(\mathbb{R}))$), and

$$\|\hat{T}_s\|_1 := \text{tr}(|\hat{T}_s|) = \frac{2}{|1-s|} \sum_{n=0}^{\infty} \left| \frac{s+1}{s-1} \right|^n = \frac{2}{|1-s| - |1+s|}, \tag{236}$$

$$\|\hat{T}_s\|_2 := \sqrt{\langle \hat{T}_s, \hat{T}_s \rangle_{\mathfrak{B}_2}} = \frac{1}{\sqrt{|\text{Re}(s)|}}; \tag{237}$$

thus, $\|\cdot\|_1$ and $\|\cdot\|_2$ are the trace class and Hilbert–Schmidt norms, respectively; moreover,

$$\text{tr}(\hat{T}_s) = 1; \quad (\text{Re}(s) < 0). \tag{238}$$

- For $\text{Re}(s) = 0$, the operator \hat{T}_s belongs to the set $(\mathcal{B}(L^2(\mathbb{R})) \setminus \mathfrak{B}_2)$.
- For $\text{Re}(s) > 0, s \neq 1$, the operator \hat{T}_s is unbounded.

Cahill and Glauber proposed the following (in general, *formal*) decomposition of a Hilbert–Schmidt operator (‘bounded’, in their terminology) $\hat{A} \in \mathfrak{B}_2$:

$$\hat{A} = \int_G \mathbf{A}_{-s}(z) \hat{T}_s(z) \frac{dz}{2\pi}, \tag{239}$$

where $\hat{T}_s(z) := U(z) \hat{T}_s U(-z), s \neq 1$, and

$$\mathbf{A}_{-s}(z) := \text{tr}(\hat{T}_{-s}(z) \hat{A}). \tag{240}$$

In particular, one can show that, for $s = 0$, formula (240)—with the trace suitably interpreted as in section 5—defines the Wigner distribution (note that $\hat{\Pi} \equiv \frac{1}{2} \hat{T}_0$ is the parity operator in $L^2(\mathbb{R})$: $(\hat{\Pi}f)(x) = f(-x)$). In general, the mathematically rigorous interpretation of the decomposition formula (239) is problematic since, for $\text{Re}(s) \neq 0$, it involves unbounded operators, either in the formula itself, or in the definition of the quasi-distribution \mathbf{A}_{-s}

⁸ We warn the reader that in the mentioned paper the terminology for indicating the bounded, Hilbert–Schmidt and trace class operators, as well as the choice of the symbols for the associated norms, is somewhat unusual.

(i.e. the pair $(\hat{T}_s, \hat{T}_{-s})$ contains an unbounded operator, for $\text{Re}(s) \neq 0$). Note, moreover, that for $s = 1$ the decomposition is not defined at all (the operator \hat{T}_1 is not defined); therefore, with the *Husimi–Kano quasi-distribution* A_{-1} (see [18, 40, 41, 51])— $A_{-1}(z) := \langle z | \hat{A} | z \rangle$, where $\{|z\rangle \equiv U(z)|0\rangle\}_{z \in \mathbb{C}}$ is the family of *coherent states* of the quantum harmonic oscillator—is not associated any (even formal) reconstruction formula.

In our framework, taking into account relation (237), with every Hilbert–Schmidt operator \hat{T}_s —with $\text{Re}(s) < 0$ —one can associate a *normalized tight frame*

$$\{\sqrt{|\text{Re}(s)|} \hat{T}_s(\mathbf{z})\}_{\mathbf{z} \in G}, \quad \text{where } \hat{T}_s(\mathbf{z}) := U(z_1) \hat{T}_s U(-z_2), \mathbf{z} \equiv (z_1, z_2), \quad (241)$$

(thus $\hat{T}_s(\mathbf{z}) \equiv \hat{T}_s(\widehat{\mathbf{z}})$), in the Hilbert space \mathfrak{B}_2 , based on (G, μ_G) . Besides, we have the decomposition formula

$$\hat{A} = \frac{|\text{Re}(s)|^{\frac{1}{2}}}{(2\pi)^2} \int_G A_s(\mathbf{z}) \hat{T}_s(\mathbf{z}) d\mathbf{z}, \quad \hat{A} \in \mathfrak{B}_2, \quad (242)$$

where

$$\begin{aligned} A_s(\mathbf{z}) &:= \sqrt{|\text{Re}(s)|} \langle \hat{T}_s(\mathbf{z}), \hat{A} \rangle_{\mathfrak{B}_2} \\ &= \sqrt{|\text{Re}(s)|} \text{tr}(\hat{T}_s(\mathbf{z})^* \hat{A}) = \sqrt{|\text{Re}(s)|} \text{tr}(\hat{T}_{s^*}(\check{\mathbf{z}}) \hat{A}), \quad \check{\mathbf{z}} \equiv (z_2, z_1). \end{aligned} \quad (243)$$

Therefore, for every $s \in \mathbb{C}$ such that $\text{Re}(s) < 0$, we have that

$$A_s(\widehat{\mathbf{z}}) = \sqrt{|\text{Re}(s)|} A_{s^*}(\mathbf{z}), \quad (244)$$

and, if \hat{A} is a trace class operator,

$$\int_G A_s(\widehat{\mathbf{z}}) d\mathbf{z} = \sqrt{|\text{Re}(s)|} \text{tr}(\hat{A}), \quad \int_G A_s(\mathbf{z}) d\mathbf{z} = \text{tr}(\hat{A}), \quad (245)$$

where we have used formula (213) and the fact that $\text{tr}(\hat{T}_s) = 1$. For $s = -1$, we have that $\hat{T}_{-1} = |0\rangle\langle 0|$; hence

$$\hat{T}_{-1}(\mathbf{z}) = |z_1\rangle\langle z_2|, \quad A_{-1}(\mathbf{z}) = \langle z_1 | \hat{A} | z_2 \rangle, \quad A_{-1}(\widehat{\mathbf{z}}) = A_{-1}(\mathbf{z}) = \langle z | \hat{A} | z \rangle, \quad (246)$$

$$\hat{A} = \frac{1}{(2\pi)^2} \int_G \langle z_1 | \hat{A} | z_2 \rangle |z_1\rangle\langle z_2| dz_1 dz_2. \quad (247)$$

Thus, the Husimi–Kano quasi-distribution A_{-1} can be regarded as the ‘restriction to the diagonal’ of the function A_{-1} , and formula (247) is the ‘non-diagonal coherent state representation of an operator’ (see [48]). Moreover, for every bounded operator $\hat{B} \in \mathcal{B}(L^2(\mathbb{R}))$, we have the following double integral decomposition (see relation (116) and remark 6)

$$\hat{B} = \frac{1}{(2\pi)^2} \int_G dz_1 \int_G dz_2 \langle z_1 | \hat{B} | z_2 \rangle |z_1\rangle\langle z_2|. \quad (248)$$

9. Conclusions and perspectives

In the present paper, we have reconsidered some fundamental aspects of the quantization–dequantization theory in the light of the mathematical notion of *frame*. We have shown (see section 2) that—in addition to the standard formulae that play a fundamental role in (generalized) wavelet analysis—by considering frames of Hilbert–Schmidt operators one is able to obtain a remarkable representation of a quantum system. It turns out that *states* (density operators) are naturally represented by ‘phase space functions’ belonging to a r.k.H.s. which is endowed with a ‘star product’; while *observables* are represented by (left and right) ‘integral

kernels' forming vector spaces endowed with a structure of Jordan–Lie algebra. Quantum-mechanical expectation values are given by simple integral formulae. We have then shown (see sections 3–5) that the classical Weyl–Wigner approach to quantization–dequantization, although not directly related to the notion of frame, relies on the notion of *square integrable projective representation*. Using this mathematical tool one can introduce (see section 6) a class of tight frames of Hilbert–Schmidt operators. A frame of this kind is generated by a square integrable representation of a group that can be regarded as the ‘symmetry group’ of a quantum system, and by an ‘analyzing operator’, whose choice can be adapted to specific applications or requirements (as it happens in wavelet analysis). Such a frame allows us to achieve a remarkable implementation (see section 7) of the abstract scheme outlined in section 2. In the case where the square integrable representation is the Weyl system, there is a link between our approach and the formalism of ‘*s*-parametrized quasi-distributions’ introduced by Cahill and Glauber (see section 8), a link that on our opinion will deserve further exploration. It is worth noticing that the *s*-parametrized quasi-distribution associated with a density operator is—for $\text{Re}(s) > 0$ —in general, a distribution (in the mathematical sense, i.e. a functional) rather than an ordinary function; this aspect has been extensively investigated in the literature in particular for the Glauber–Sudarshan ‘*P* quasi-distribution’ (corresponding to the case where $s = 1$), see [23, 48] and references therein. On the contrary, in our framework, with both states and observables are associated ordinary functions: square integrable functions and integral kernels, respectively. This is coherent with the distinct roles that these physical entities play in quantum theory.

We plan to develop the basic results established in the present contribution in several directions. In particular, we will mention the representation—in our framework—of specific quantum systems and of ‘super-operators’ (that play a fundamental role in the theory of open quantum systems), and the study of the classical limit of quantum mechanics. Finally, we note that our approach and results may turn to be relevant for the important issue of *informational completeness* (see [52, 53] and references therein) in quantum mechanics.

Acknowledgments

V I Man’ko wishes to thank the University of Napoli ‘Federico II’ and the INFN (Sezione di Napoli) for the kind hospitality, and the Russian Foundation for Basic Research (under Project no 07-02-00598) for partial support. The main results of the paper were presented by one of the authors (PA) at the conference *Problemi Attuali di Fisica Teorica* (14–19 March 2008, Vietri sul Mare, Italy). He wishes to thank the organizers for their very kind hospitality.

References

- [1] Wigner E 1932 On the quantum correction for thermodynamic equilibrium *Phys. Rev.* **40** 749–59
- [2] Weyl H 1950 *The Theory of Groups and Quantum Mechanics* (New York: Dover)
- [3] Zachos C K, Fairlie D B and Curtright T L (ed) 2005 *Quantum Mechanics in Phase Space* (Singapore: World Scientific)
- [4] Man’ko O V, Man’ko V I and Marmo G 2002 Alternative commutation relations, star products and tomography *J. Phys. A: Math. Gen.* **35** 699–719
- [5] Man’ko V I, Marmo G, Sudarshan E C G and Zaccaria F 2004 The geometry of density states, positive maps and tomograms *Symmetries in Science XI* ed B J Gruber, G Marmo and N Yoshinaga (Dordrecht: Kluwer)
- [6] Man’ko O V, Man’ko V I, Marmo G and Vitale P 2007 Star products, duality and double Lie algebras *Phys. Lett. A* **360** 522–32
- [7] D’Ariano G M, Maccone L and Painsi M 2003 Spin tomography *J. Opt. B: Quantum Semiclass. Opt.* **5** 77–84
- [8] D’Ariano G M and Sacchi M F 2005 Characterization of tomographically faithful states in terms of their Wigner function *J. Opt. B: Quantum Semiclass. Opt.* **7** S408–12

- [9] Chiribella G, D'Ariano G M and Perinotti P 2006 Applications of the group $SU(1, 1)$ for quantum computation and tomography *Laser Phys.* **16** 1572–81
- [10] Duffin R J and Schaeffer A C 1952 A class of nonharmonic Fourier series *Trans. Am. Math. Soc.* **72** 341–66
- [11] Ali S T, Antoine J P, Gazeau J P and Mueller U A 1995 Coherent states and their generalizations: a mathematical overview *Rev. Math. Phys.* **7** 1013–104
- [12] Ali S T, Antoine J P and Gazeau J P 2000 *Coherent States, Wavelets and Their Generalizations* (Berlin: Springer)
- [13] Daubechies I 1992 *Ten Lectures on Wavelets* (Philadelphia: SIAM)
- [14] Kaiser G 1994 *A Friendly Guide to Wavelets* (Basle: Birkhäuser)
- [15] Man'ko M A, Man'ko V I and Mendes R V 2001 Tomograms and other transforms: a unified view *J. Phys. A: Math. Gen.* **34** 8321
- [16] Klauder J R 1963 Continuous representation theory: I. Postulates of continuous representation theory *J. Math. Phys.* **4** 1055–8
Klauder J R 1963 Continuous representation theory: II. Generalized relation between quantum and classical dynamics *J. Math. Phys.* **4** 1058–73
Klauder J R 1964 Continuous representation theory: III. On functional quantization of classical systems *J. Math. Phys.* **5** 177–87
Klauder J R 1964 Continuous representation theory: IV. Structure of a class of function spaces arising from quantum mechanics *J. Math. Phys.* **5** 878–96
- [17] Aslaksen E W and Klauder J R 1969 Continuous representation theory using the affine group *J. Math. Phys.* **10** 2267–75
- [18] Cahill K E and Glauber R J 1969 Ordered expansions in boson amplitude operators *Phys. Rev.* **177** 1857–81
Cahill K E and Glauber R J 1969 Density operators and quasiprobability distributions *Phys. Rev.* **177** 1882–902
- [19] Duflo M and Moore C C 1976 On the regular representation of a nonunimodular locally compact group *J. Funct. Anal.* **21** 209–43
- [20] Grossmann A, Morlet J and Paul T 1985 Integral transforms associated to square integrable representations I. General results *J. Math. Phys.* **26** 2473–9
Grossmann A, Morlet J and Paul T 1986 Integral transforms associated to square integrable representations: II. Examples *Ann. Inst. Henri Poincaré* **45** 293–309
- [21] Aniello P, Cassinelli G, De Vito E and Levrero A 1998 Square-integrability of induced representations of semidirect products *Rev. Math. Phys.* **10** 301–13
- [22] Aniello P 2006 Square integrable projective representations and square integrable representations modulo a relatively central subgroup *Int. J. Geom. Meth. Mod. Phys.* **3** 233–67
- [23] Perelomov A 1986 *Generalized Coherent States and Their Applications* (Berlin: Springer)
- [24] Schrödinger E 1926 Der stetige Übergang von der Mikro- zur Makromechanik *Naturwissenschaften* **14** 664–6
- [25] Glauber R J 1963 The quantum theory of optical coherence *Phys. Rev.* **130** 2529–39
- [26] Sudarshan E C G 1963 Equivalence of semiclassical and quantum mechanical description of statistical light beams *Phys. Rev. Lett.* **10** 277–9
- [27] Aniello P, Cassinelli G, De Vito E and Levrero A 1998 Wavelet transforms and discrete frames associated to semidirect products *J. Math. Phys.* **39** 3965–73
- [28] Aniello P, Cassinelli G, De Vito E and Levrero A 2001 On discrete frames associated with semidirect products *J. Fourier Anal. Appl.* **7** 199–206
- [29] Folland G B 1989 *Harmonic Analysis in Phase Space* (Princeton, NJ: Princeton University Press)
- [30] Wong M W 1998 *Weyl Transforms* (Berlin: Springer)
- [31] Aronszajn N 1950 Theory of reproducing kernels *Trans. Am. Math. Soc.* **68** 337–404
- [32] Hille E 1972 Introduction to general theory of reproducing kernels *Rocky Mt. J. Math.* **2** 321–68
- [33] Saitoh S 1997 *Integral Transforms, Reproducing Kernels and Their Applications* (Harlow: Addison Wesley Longman)
- [34] Ambrose W 1945 Structure theorems for a certain class of Banach algebras *Trans. Am. Math. Soc.* **57** 364–86
- [35] Reed M and Simon B 1972 *Methods of Modern Mathematical Physics I: Functional Analysis* (New York: Academic)
- [36] Grossmann A, Loupias G and Stein E M 1968 An algebra of pseudo-differential operators and quantum mechanics in phase space *Ann. Inst. Fourier* **18** 343–68
- [37] Berezin F A 1975 General concept of quantization *Commun. Math. Phys.* **40** 153–74
- [38] Bayen F, Flato M, Fronsdal C, Lichnerowicz A and Sternheimer D 1978 Deformation theory and quantization. I. Deformations of symplectic structures *Ann. Phys.* **111** 61–110
Bayen F, Flato M, Fronsdal C, Lichnerowicz A and Sternheimer D 1978 Deformation theory and quantization. II. Physical applications *Ann. Phys.* **111** 111–51

- [39] Emch G 1984 *Mathematical and Conceptual Foundations of 20th-Century Physics* (Amsterdam: North-Holland)
- [40] Aniello P, Man'ko V I, Marmo G, Solimeno S and Zaccaria F 2000 On the coherent states, displacement operators and quasidistributions associated with deformed quantum oscillators *J. Opt. B: Quantum Semiclass. Opt.* **2** 718–25
- [41] Schleich W P 2001 *Quantum Optics in Phase Space* (New York: Wiley-VCH)
- [42] Man'ko V I, Marmo G, Simoni A, Sudarshan E C G and Ventriglia F 2008 A tomographic setting for quasi-distribution functions *Rep. Math. Phys.* **61** 337–59
- [43] Klauder J R and Skagerstam B K 2007 Generalized phase-space representation of operators *J. Phys. A: Math. Theor.* **40** 2093–105
- [44] Hillery M, O'Connell R, Scully M and Wigner E 1984 Distribution functions in physics: fundamentals *Phys. Rep.* **106** 121–67
- [45] Esposito G, Marmo G and Sudarshan G 2004 *From Classical to Quantum Mechanics* (Cambridge: Cambridge University Press)
- [46] Varadarajan V S 1985 *Geometry of Quantum Theory* 2nd edn (Berlin: Springer)
- [47] Folland G B 1995 *A Course in Abstract Harmonic Analysis* (Boca Raton, FL: CRC Press)
- [48] Klauder J R and Sudarshan E C G 1968 *Fundamentals of Quantum Optics* (New York: Benjamin)
- [49] Segal I E 1950 The two-sided regular representation of a unimodular locally compact group *Ann. Math.* **51** 293–8
- [50] Pool J C T 1966 Mathematical aspects of Weyl correspondence *J. Math. Phys.* **7** 66–76
- [51] Kano Y 1965 A new phase-space distribution function in the statistical theory of the electromagnetic field *J. Math. Phys.* **6** 1913–5
- [52] Prugovecki E 1977 Information-theoretical aspects of quantum measurement *Int. J. Theor. Phys.* **16** 321–31
- [53] Busch P, Grabowski M and Lahti P J 1997 *Operational Quantum Physics* (Berlin: Springer)